Introduction to Supersymmetry

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Abstract

These are expanded notes of lectures given at the summer school "Gif 2000" in Paris. They constitute the first part of an "Introduction to supersymmetry and supergravity" with the second part on supergravity by J.-P. Derendinger to appear soon.

The present introduction is elementary and pragmatic. I discuss: spinors and the Poincaré group, the susy algebra and susy multiplets, superfields and susy lagrangians, susy gauge theories, spontaneously broken susy, the non-linear sigma model, N=2 susy gauge theories, and finally Seiberg-Witten duality.

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Chapter 1

Introduction

Supersymmetry not only has played a most important role in the development of theoretical physics over the last three decades, but also has strongly influenced experimental particle physics.

Supersymmetry first appeared in the early seventies in the context of string theory where it was a symmetry of the *two-dimensional* world sheet theory. At this time it was more considered as a purely theoretical tool. Shortly after it was realised that supersymmetry could be a symmetry of four-dimensional quantum field theories and as such could well be directly relevant to elementary particle physics. String theories with supersymmetry on the world-sheet, if suitably modified, were shown to actually exhibit supersymmetry in space-time, much as the four-dimensional quantum field theories: this was the birth of superstrings. Since then, countless supersymmetrc theories have been developed with minimal or extended global supersymmetry or with a local version of supersymmetry which is supergravity.

There are several reasons why an elementary particle physicist wants to consider supersymmetric theories. The main reason is that radiative corrections tend to be less important in supersymmetric theories, due to cancellations between fermion loops and boson loops. As a result certain quantities that are small or vanish classically (i.e. at tree level) will remain so once radiative corrections (loops) are taken into account. Famous examples include the vanishing or extreme smallness of the cosmological constant, the hierarchy problem (why is there such a big gap between the Planck scale / GUT scale and the scale of electroweak symmetry breaking) or the issue of renormalisation of quantum gravity. While supersymmetry could solve most if not all of these questions, it cannot be the full answer, since we know that supersymmetry cannot be exactly realised in nature: it must be broken at experimentally accessible energies since otherwise one certainly would have detected many of the additional particles it predicts.

Supersymmetric models often are easier to solve than non-supersymmetric ones since they are more constrained by the higher degree of symmetry. Thus they may serve as toy models where certain analytic results can be obtained and may serve as a qualitative guide to the behaviour of more realistic theories. For example the study of supersymmetric versions of QCD have given quite some insights in the strong coupling dynamics responsible for phenomena like quark confinement. In this type of studies the basic property is a duality (a mapping) between a weakly and a strongly coupled theory. It seems that dualities are difficult to realise in non-supersymmetric theories but are rather easily present in supersymmetric ones. The study of dualities in superstring theories has been particular fruitful over the last five years or so.

Supersymmetry has also appeared outside the realm of elementary particle physics and has found applications in condensed matter systems, in particular in the study of disordered systems.

In these lectures, I will try to give an elementary and pragmatic introduction to supersymmetry. In the first four chapters, I introduce the supersymmetry algebra and its basic representations, i.e. the supermultiplets and then present supersymmetric field theories with emphasis on supersymmetric gauge theories. The presentation is pragmatic in the sense that I try to introduce only as much mathematical structure as is necessary to arrive at the supersymmetric field theories. No emphasis is put on uniqueness theorems or the like. On the other hand, I very quickly introduce superspace and superfields as a useful tool because it allows to easily and efficiently construct supersymmetric Lagrangians. The discussion remains classical and due to lack of time the issue of renormalisation is not discussed here. Then follows a brief discussion of spontaneous breaking of supersymmetry. The supersymmetric non-linear sigma model is discussed in some detail as it is relevant to the coupling of supergravity to matter multiplets. Finally I focus on $N = 2$ extended supersymmetric gauge theories followed by a rather detailed introduction to the determination of their low-energy effective action, taking advantage of duality and the rigid mathematical structure of $N = 2$ supersymmetry.

There are many textbooks and review articles on supersymmetry (see e.g. [1] to [8]) that complement the present introduction and also contain many references to the original literature which are not given here.

Chapter 2

Spinors and the Poincaré group

We begin with a review of the Lorentz and Poincaré groups and spinors in four-dimensional Minkowski space. The signature is taken to be +,-,-,- so that $p^2 = +m^2$ and $\mu, \nu, \ldots$ always are space-time indices, while $i, j, \ldots$ are only space indices. Then the metric $g_{\mu\nu}$ is diagonal with $g_{00} = 1$, $g_{11} = g_{22} = g_{33} = -1$.

2.1 The Lorentz and Poincaré groups

The Lorentz group has six generators, three rotations J_i and three boosts K_i, $i = 1, 2, 3$ with commutation relations

$$[J_i, J_j] = i\epsilon_{ijk} J_k \ , \quad [K_i, K_j] = -i\epsilon_{ijk} J_k \ , \quad [J_i, K_j] = i\epsilon_{ijk} K_j \ . \tag{2.1}$$

To identify the mathematical structure and to construct representations of this algebra one introduces the linear combinations

$$J_j^{\pm} = \frac{1}{2}(J_j \pm iK_j) \tag{2.2}$$

in terms of which the algebra separates into two commuting SU(2) algebras:

$$[J_i^{\pm}, J_j^{\pm}] = i\epsilon_{ijk} J_k^{\pm} \ , \quad [J_i^{\pm}, J_j^{\mp}] = 0 \ . \tag{2.3}$$

These generators are not hermitian however, and we see that the Lorentz group is a complexified version of $\mathrm{SU}(2) \times \mathrm{SU}(2)$: this group is $\mathrm{Sl}(2, \mathbf{C})$. (More precisely, $\mathrm{Sl}(2, \mathbf{C})$ is the universal cover of the Lorentz group, just as SU(2) is the universal cover of SO(3).) To see that this group is really $\mathrm{Sl}(2, \mathbf{C})$ is easy: introduce the four 2×2 matrices σ_μ where σ_0 is the identity matrix and σ_i, $i = 1, 2, 3$ are the three Pauli matrices. (Note that we always write the Pauli matrices with a lower index i, while $\sigma^0 = \sigma_0$ and $\sigma^i = -\sigma_i$.) Then for every four-vector x^μ the 2×2 matrix $x^\mu \sigma_\mu$ is hermitian and has determinant equal to $x^\mu x_\mu$ which is a Lorentz invariant. Hence a Lorentz transformation preserves the determinant and the hermiticity of this matrix, and thus must act as $x^\mu \sigma_\mu \to A x^\mu \sigma_\mu A^\dagger$ with $|\det A| = 1$. We see that up to an irrelevant phase, A is a complex 2×2 matrix of unit determinant, i.e. an element of $\mathrm{Sl}(2, \mathbf{C})$. This establishes the mapping between an element of the Lorentz group and the group $\mathrm{Sl}(2, \mathbf{C})$.

The Poincaré group contains, in addition to the Lorentz transformations, also the translations. More precisely it is a semi-direct product of the Lorentz-group and the group of translations in space-time. The generators of the translations are usually denoted P_μ. In addition to the commutators of the Lorentz generators J_i (rotations) and K_i (boosts) one has the following commutation relations involving the P_μ:

$$
\begin{aligned}
[P_\mu, P_\nu] &= 0 \ , \\
[J_i, P_j] &= i\epsilon_{ijk} P_k \ , \ [J_i, P_0] = 0 \ , \ [K_i, P_j] = -iP_0 \ , \ [K_i, P_0] = -iP_j \ ,
\end{aligned}
\tag{2.4}
$$

which state that translations commute among themselves, that the P_i are a vector and P_0 a scalar under space rotations and how P_i and P_0 mix under a boost. The Lorentz and Poincaré algebras are often written in a more covariant looking, but less intuitive form. One defines the Lorentz generators $M_{\mu\nu} = -M_{\nu\mu}$ as $M_{0i} = K_i$ and $M_{ij} = \epsilon_{ijk} J_k$. Then the full Poincaré algebra reads

$$
\begin{aligned}
[P_\mu, P_\nu] &= 0 \ , \\
[M_{\mu\nu}, M_{\rho\sigma}] &= ig_{\nu\rho} M_{\mu\sigma} - ig_{\mu\rho} M_{\nu\sigma} - ig_{\nu\sigma} M_{\mu\rho} + ig_{\mu\sigma} M_{\nu\rho} \ , \\
[M_{\mu\nu}, P_\rho] &= -ig_{\rho\mu} P_\nu + ig_{\rho\nu} P_\mu \ .
\end{aligned}
\tag{2.5}
$$

2.2 Spinors

Two-component spinors

There are various equivalent ways to introduce spinors. Here we define spinors as the objects carrying the basic representation of $\mathrm{Sl}(2, \mathbf{C})$. Since elements of $\mathrm{Sl}(2, \mathbf{C})$ are complex 2×2 matrices, a spinor is a two complex component object $\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$ transforming under an element $\mathcal{M} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \mathrm{Sl}(2, \mathbf{C})$ as

$$
\psi_\alpha \to \psi'_\alpha = \mathcal{M}_\alpha^{\ \beta} \ \psi_\beta \ , \tag{2.6}
$$

with $\alpha, \beta = 1, 2$ labeling the components. Now, unlike for SU(2), for $\mathrm{Sl}(2, \mathbf{C})$ a representation and its complex conjugate are not equivalent. $\mathcal{M}$ and $\mathcal{M}^*$ give inequivalent representations. A two-component object $\overline{\psi}$ transforming as

$$\overline{\psi}_{\dot\alpha} \to \overline{\psi}'_{\dot\alpha} = \mathcal{M}^{*}{}_{\dot\alpha}^{\;\dot\beta}\,\overline{\psi}_{\dot\beta} \tag{2.7}$$

is called a dotted spinor, while the above ψ is called an undotted one. Comparing the complex conjugate of (2.6) with (2.7) we see that we can identify $\overline{\psi}_{\dot\alpha}$ with $(\psi_\alpha)^*$.

The representation carried by the ψ_α is called $(\frac{1}{2},0)$ (matrices $\mathcal{M}$) and the one carried by the $\overline{\psi}_{\dot\alpha}$ is called $(0,\frac{1}{2})$ (matrices $\mathcal{M}^*$). They are both irreducible. Now, any $\mathrm{Sl}(2,\mathbf{C})$ matrix can be written as

$$\begin{aligned} \mathcal{M} &= \exp(a_j\sigma_j + ib_j\sigma_j) \\ \mathcal{M}^* &= \exp(a_j\sigma_j^* - ib_j\sigma_j^*)\ . \end{aligned} \tag{2.8}$$

This explicitly displays the generators as the spin $\frac{1}{2}$ representation of the complexified SU(2), in accordance with (2.2).

It proves very useful to now introduce some notations and conventions. We intoduce the antisymmetric two-index tensors $\epsilon^{\alpha\beta}$ and $\epsilon_{\alpha\beta}$

$$\epsilon^{\alpha\beta} = \epsilon^{\dot\alpha\dot\beta} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\ , \quad \epsilon_{\alpha\beta} = \epsilon_{\dot\alpha\dot\beta} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tag{2.9}$$

which are used to raise and lower indices as follows:

$$\psi^\alpha = \epsilon^{\alpha\beta}\psi_\beta\ , \quad \psi_\alpha = \epsilon_{\alpha\beta}\psi^\beta\ , \quad \overline{\psi}^{\dot\alpha} = \epsilon^{\dot\alpha\dot\beta}\overline{\psi}_{\dot\beta}\ , \quad \overline{\psi}_{\dot\alpha} = \epsilon_{\dot\alpha\dot\beta}\overline{\psi}^{\dot\beta}\ . \tag{2.10}$$

One can then easily show that the transformation under an element $\mathcal{M}$ of $\mathrm{Sl}(2,\mathbf{C})$ is $\psi'^\alpha = \psi^\beta(\mathcal{M}^{-1})_\beta^{\ \alpha}$ and $\overline{\psi}'^{\dot\alpha} = \overline{\psi}^{\dot\beta}(\mathcal{M}^{*-1})_{\dot\beta}^{\ \dot\alpha}$.

The four σ_μ matrices introduced above naturally have a dotted and an undotted index. Recalling that our signature is +,-,-,- we have

$$(\sigma^\mu)_{\alpha\dot\alpha} = (\mathbf{1}, -\sigma_i)_{\alpha\dot\alpha}\ . \tag{2.11}$$

Raising the indices using the ϵ tensors yields

$$(\overline{\sigma}^\mu)^{\dot\alpha\alpha} = \epsilon^{\dot\alpha\dot\beta}\epsilon^{\alpha\beta}(\sigma^\mu)_{\beta\dot\beta} = (\mathbf{1}, +\sigma_i)^{\dot\alpha\alpha}\ . \tag{2.12}$$

Whenever we consider expressions involving more than one spinor we have to remember that spinors anticommute. Hence (with two-component spinors) $\psi_1\chi_2 = -\chi_2\psi_1$, as well as $\psi_1\overline{\chi}_{\dot 2} = -\chi_{\dot 2}\psi_1$ etc. The scalar products $\psi\chi$ and $\overline{\psi}\overline{\chi}$ are defined as

$$\begin{aligned} \psi\chi \equiv \psi^\alpha\chi_\alpha &= \epsilon^{\alpha\beta}\psi_\beta\chi_\alpha = -\epsilon^{\alpha\beta}\psi_\alpha\chi_\beta = -\psi_\alpha\chi^\alpha = \chi^\alpha\psi_\alpha = \chi\psi \\ \overline{\psi}\overline{\chi} \equiv \overline{\psi}_{\dot\alpha}\overline{\chi}^{\dot\alpha} &= \ldots = \overline{\chi}_{\dot\alpha}\overline{\psi}^{\dot\alpha} \\ (\psi\chi)^\dagger &= \overline{\chi}_{\dot\alpha}\overline{\psi}^{\dot\alpha} = \overline{\chi}\overline{\psi} = \overline{\psi}\overline{\chi}\ . \end{aligned} \tag{2.13}$$

Note that by convention undotted indices are always contracted from upper left to lower right, while dotted indices are always contracted from lower left to upper right. Note however that this rule does not apply when rising or lowering spinor indices with the ϵ-tensor. With this rule we also have

$$\psi\sigma^\mu\overline{\chi} = \psi^\alpha\sigma^\mu_{\alpha\dot{\beta}}\overline{\chi}^{\dot{\beta}} \quad , \quad \overline{\psi}\overline{\sigma}^\mu\chi = \overline{\psi}_{\dot{\alpha}}\overline{\sigma}^{\mu\dot{\alpha}\beta}\chi_\beta \ . \tag{2.14}$$

One can then prove a certain amount of useful identities which we summarise here:

$$\begin{aligned} \chi\sigma^\mu\overline{\psi} = -\overline{\psi}\overline{\sigma}^\mu\chi \quad &, \quad \chi\sigma^\mu\overline{\sigma}^\nu\psi = \psi\sigma^\nu\overline{\sigma}^\mu\chi \\ (\chi\sigma^\mu\overline{\psi})^\dagger = \psi\sigma^\mu\overline{\chi} \quad &, \quad (\chi\sigma^\mu\overline{\sigma}^\nu\psi)^\dagger = \overline{\psi}\overline{\sigma}^\nu\sigma^\mu\overline{\chi} \\ \psi\chi = \chi\psi \quad , \quad \overline{\psi}\overline{\chi} \ &= \ \overline{\chi}\overline{\psi} \quad , \quad (\psi\chi)^\dagger = \overline{\psi}\overline{\chi} \ . \end{aligned} \tag{2.15}$$

<u>Dirac spinors</u>

One introduces the Dirac matrices in the Weyl representation as

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \overline{\sigma}^\mu & 0 \end{pmatrix} \ , \quad \gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} \mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{pmatrix} \tag{2.16}$$

A four-component Dirac spinor is made from a two-component undotted and a two-component dotted spinor as $\begin{pmatrix} \psi_\alpha \\ \overline{\chi}^{\dot{\alpha}} \end{pmatrix}$. Clearly it transforms as the reducible $(\frac{1}{2},0)\oplus(0,\frac{1}{2})$ representation of the Lorentz group. Then $\begin{pmatrix} \psi_\alpha \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ \overline{\chi}^{\dot{\alpha}} \end{pmatrix}$ are chiral Dirac (or Weyl) spinors. A Majorana spinor is a Dirac spinor with $\chi \equiv \psi$, i.e. it is of the form $\begin{pmatrix} \psi_\alpha \\ \overline{\psi}^{\dot{\alpha}} \end{pmatrix}$. The Lorentz generators are

$$\Sigma^{\mu\nu} = \frac{i}{2}\gamma^{\mu\nu} \ , \quad \gamma^{\mu\nu} = \frac{1}{2}(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu) = \frac{1}{2}\begin{pmatrix} \sigma^\mu\overline{\sigma}^\nu - \sigma^\nu\overline{\sigma}^\mu & 0 \\ 0 & \overline{\sigma}^\mu\sigma^\nu - \overline{\sigma}^\nu\sigma^\mu \end{pmatrix} \ . \tag{2.17}$$

We see that indeed the undotted and dotted spinors transform separately, the generators being $i\sigma^{\mu\nu}$ for ψ_α and $i\overline{\sigma}^{\mu\nu}$ for $\overline{\psi}^{\dot{\alpha}}$ with

$$\begin{aligned} (\sigma^{\mu\nu})_\alpha^{\ \beta} \ &= \ \tfrac{1}{4}\left(\sigma^\mu_{\alpha\dot{\gamma}}\overline{\sigma}^{\nu\dot{\gamma}\beta} - (\mu\leftrightarrow\nu)\right) \\ (\overline{\sigma}^{\mu\nu})^{\dot{\alpha}}_{\ \dot{\beta}} \ &= \ \tfrac{1}{4}\left(\overline{\sigma}^{\mu\dot{\alpha}\gamma}\sigma^\nu_{\gamma\dot{\beta}} - (\mu\leftrightarrow\nu)\right) \ . \end{aligned} \tag{2.18}$$

Note that e.g. $\sigma^{12} = \overline{\sigma}^{12} = -\frac{i}{2}\sigma_3 \equiv -\frac{i}{2}\sigma_z$ so that the rotation generator $M_{12} = M^{12}$ is $\frac{1}{2}\sigma_z$ as expected.

<u>Casimirs: mass and helicity</u>

A useful quantity is the Pauli-Lubanski vector

$$W^\mu = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}P_\nu M_{\rho\sigma} \ , \tag{2.19}$$

which can be easily shown to commute with the P_μ and behaves as a four-vector under commutation with the Lorentz generators. It follows that $W^2 \equiv W^\mu W_\mu$ as well as $P^2 \equiv P_\mu P^\mu$ commute with all the generators, i.e. they are two (and the only two) Casimirs of the Poincaré group. For a massive particle one can go to the rest frame where $P_\mu = (m, 0, 0, 0)$ and then $P^2 = m^2$ and $W^2 = -m^2 s(s+1)$ where s is the spin of the particle. The different states of this irreducible representation are distinguished by the value of p_i and of $M_{12} \equiv J_3 = S_3$ in the rest frame because in the rest frame only the spin and not the orbital part contributes to the angular momentum. In the above representation of dotted or undotted spinors one has of course $s = \frac{1}{2}$. For a massless particle $P^2 = 0$ and also $W^2 = 0$. We may take $P_\mu = (E, 0, 0, E)$ so that $W^\mu = M_{12} P^\mu$ with $M_{12} = \pm s$ being the helicity. For such massless particles s is fixed and the different states of this irreducible representation are distinguished by the sign of the helicity and by the values of p_i.

Chapter 3

The susy algebra and its representations

3.1 The supersymmetry algebra

We want to enlarge the Poincaré algebra by generators that transform either as undotted spinors Q^I_α or as dotted spinors $\overline{Q}^I_{\dot\alpha}$ under the Lorentz group and that commute with the translations. The extra index $I = 1, \ldots N$ labels the different spinorial generators in case there are more than one pair. This means that according to (2.18)

$$\begin{aligned} [P_\mu, Q^I_\alpha] &= 0 \ , \\ [P_\mu, \overline{Q}^I_{\dot\alpha}] &= 0 \ , \\ [M_{\mu\nu}, Q^I_\alpha] &= i(\sigma_{\mu\nu})_\alpha{}^\beta Q^I_\beta , \\ [M_{\mu\nu}, \overline{Q}^{I\dot\alpha}] &= i(\overline{\sigma}_{\mu\nu})^{\dot\alpha}{}_{\dot\beta} \overline{Q}^{I\dot\beta} \ . \end{aligned} \tag{3.1}$$

In particular, $M_{12} \equiv J_3$ and thus $[J_3, Q^I_1] = \frac{1}{2} Q^I_1$ and $[J_3, Q^I_2] = -\frac{1}{2} Q^I_2$. Since $\overline{Q}^{I1} = -(Q^I_2)^\dagger$ and $\overline{Q}^{I2} = (Q^I_1)^\dagger$ one similarly has $[J_3, (Q^I_2)^\dagger] = \frac{1}{2}(Q^I_2)^\dagger$ and $[J_3, (Q^I_1)^\dagger] = -\frac{1}{2}(Q^I_1)^\dagger$. We conclude that Q^I_1 and $(Q^I_2)^\dagger$ rise the z-component of the spin (helicity) by half a unit, while Q^I_2 and $(Q^I_1)^\dagger$ lower it by half a unit.

Since the Q^I_α transform in the $(\frac{1}{2}, 0)$ representation and the $\overline{Q}^I_{\dot\alpha}$ in the $(0, \frac{1}{2})$, the anticommutator of Q^I_α and $\overline{Q}^J_{\dot\beta}$ must transform as $(\frac{1}{2}, \frac{1}{2})$, i.e. as a four vector. The obvious candidate is P_μ so that we arrive at

$$\{Q^I_\alpha, \overline{Q}^J_{\dot\beta}\} = 2\sigma^\mu_{\alpha\dot\beta} P_\mu \delta^{IJ} \ . \tag{3.2}$$

The δ^{IJ} can always be achieved by diagonalising an a priori arbitrary symmetric matrix and by rescaling the Q and $\overline{Q}$. Furthermore, since $\overline{Q}$ is the adjoint of Q, positivity of the Hilbert space excludes zero eigenvalues of this matrix. Finally

$$\{Q^I_\alpha, Q^J_\beta\} = \epsilon_{\alpha\beta} Z^{IJ} \ , \quad \{\overline{Q}^I_{\dot\alpha}, \overline{Q}^J_{\dot\beta}\} = \epsilon_{\dot\alpha\dot\beta} (Z^{IJ})^* \ . \tag{3.3}$$

The $Z^{IJ} = -Z^{JI}$ are central charges which means they commute with all generators of the full algebra. The simplest algebra has $N = 1$, i.e. there are no indices I, J and there is no possibility of central charges. This is the unextended susy algebra. If $N > 1$ one talks about extended supersymmetry. In the simplest extended case, $N = 2$, there is just one central charge $Z \equiv Z^{12}$. From the algebraic point of view there is no limit on N, but we will see that with increasing N the theories also must contain particles of increasing spin and there seem to be no consistent quantum field theories with spins larger than one (without gravity) or larger than two (with gravity) leading to $N \leq 4$, resp. $N \leq 8$.

3.2 Some basic properties

Using the above susy algebra it is easy to establish some basic properties of supersymmetric theories. Since the full susy algebra contains the Poincaré algebra as a subalgebra, any representation of the full susy algebra also gives a representation of the Poincaré algebra, although in general a reducible one. Since each irreducible representation (of the type considered above) of the Poincaré algebra corresponds to a particle, an irreducible representation of the susy algebra in general corresponds to several particles. The corresponding states are related to each other by the Q^I_α and $\overline{Q}^J_{\dot\beta}$ and thus have spins differing by units of one half. They form a supermultiplet. By abuse of language we will call an irreducioble representation of the susy algebra simply a supermultiplet. Clearly, using the spin-statistics theorem, the Q and $\overline{Q}$ change bosons into fermions and vice versa. One then has:

All particles belonging to an irreducible representation of susy, i.e. within one supermultiplet, have the same mass. This is obvious since P^2 commutes with all generators of the susy algebra, i.e. it is still a Casimir operator.

In a supersymmetric theory the energy P_0 is always positive. To see this, let $|\Phi\rangle$ be any state. Then by the positivity of the Hilbert space we have

$$\begin{aligned} 0 \ &\leq \ ||Q^I_\alpha \,|\Phi\rangle\,||^2 + ||(Q^I_\alpha)^\dagger \,|\Phi\rangle\,||^2 = \langle\Phi| \left((Q^I_\alpha)^\dagger Q^I_\alpha + Q^I_\alpha (Q^I_\alpha)^\dagger \,|\Phi\rangle \right) \\ &= \ \langle\Phi|\, \{Q^I_\alpha, \overline{Q}^I_{\dot\alpha}\}\,|\Phi\rangle = 2\sigma^\mu_{\alpha\dot\alpha}\,\langle\Phi|\,P_\mu\,|\Phi\rangle \end{aligned} \tag{3.4}$$

since $(Q^I_\alpha)^\dagger \equiv \overline{Q}^I_{\dot\alpha}$. Summing this over $\alpha \equiv \dot\alpha = 1, 2$ and using $\mathrm{tr}\,\sigma^\mu = 2\delta^{\mu 0}$ yields $0 \leq 4\,\langle\Phi|\,P_0\,|\Phi\rangle$, which was to be shown.

A supermultiplet always contains an equal number of bosonic and fermionic degrees of freedom. By degrees of freedom one means physical (positive norm) states. Hence a photon has two degrees of freedom corresponding to the two helicities +1 and −1 (the two polarizations). Let the fermion number be N_F equal one on a fermionic state and 0 on a bosonic one. Equivalently $(-)^{N_F}$ is +1 on bosons and −1 on fermions. We want to show that

$$\mathrm{Tr}\,(-)^{N_F} = 0 \tag{3.5}$$

if the trace is taken over any finite-dimensional representation. Note that $(-)^{N_F}$ anticommutes with Q. Using the cyclicity of the trace, one has

$$\begin{aligned} 0 &= \mathrm{Tr}\left(-Q_\alpha(-)^{N_F}\overline{Q}_{\dot\beta} + (-)^{N_F}\overline{Q}_{\dot\beta}Q_\alpha\right) = \mathrm{Tr}\left((-)^{N_F}\{Q_\alpha, \overline{Q}_{\dot\beta}\}\right) \\ &= 2\sigma^\mu_{\alpha\dot\beta}\mathrm{Tr}\left((-)^{N_F}P_\mu\right) \ . \end{aligned} \tag{3.6}$$

Choosing any non-vanishing momentum P_μ gives the desired result.

3.3 Massless supermultiplets

We will first assume that all central charges Z^{IJ} vanish. Below we will see that for massless representations this is necessarily a consequence of the positivity of the Hilbert space. Then all Q^I_α anticommute among themselves, and so do the $\overline{Q}^J_{\dot\beta}$. Since $P^2 = 0$ we choose a reference frame where $P_\mu = E(1,0,0,1)$ so that $\sigma^\mu P_\mu = \begin{pmatrix} 0 & 0 \\ 0 & 2E \end{pmatrix}$ and thus

$$\{Q^I_\alpha, \overline{Q}^J_{\dot\beta}\} = \begin{pmatrix} 0 & 0 \\ 0 & 4E \end{pmatrix}_{\alpha\dot\beta} \delta^{IJ} \ . \tag{3.7}$$

In particuler, $\{Q^I_1, \overline{Q}^J_{\dot 1}\} = 0,\ \forall I, J$. On a positive definite Hilbert space we must then set $Q^I_1 = \overline{Q}^J_{\dot 1} = 0,\ \forall I, J$. The argument is similar to the one above:

$$0 = \langle\Phi|\,\{Q^I_1, \overline{Q}^I_{\dot 1}\}\,|\Phi\rangle = ||Q^I_1\,|\Phi\rangle\,||^2 + ||\overline{Q}^I_{\dot 1}\,|\Phi\rangle\,||^2 \Rightarrow Q^I_1 = \overline{Q}^J_{\dot 1} = 0 \tag{3.8}$$

Thus we are left with only the Q^I_2 and $\overline{Q}^J_{\dot 2}$, i.e. N of the initial $2N$ fermionic generators. If we define

$$a_I = \frac{1}{\sqrt{4E}}Q^I_2 \ , \quad a^\dagger_I = \frac{1}{\sqrt{4E}}\overline{Q}^I_{\dot 2} \tag{3.9}$$

the a_I and $a^\dagger_I$ are anticommuting annihilation and creation operators:

$$\{a_I, a^\dagger_J\} = \delta_{IJ} \ , \quad \{a_I, a_J\} = \{a^\dagger_I, a^\dagger_J\} = 0 \ . \tag{3.10}$$

One then chooses a "vacuum state", i.e. a state annihilated by all the a_I. Such a state will carry some irreducible representation of the Poincaré algebra, i.e. in addition to its zero mass it is characterised by some helicity λ_0. We denote this state as $|\lambda_0\rangle$. From the commutators of Q_2^I and $\overline{Q}_2^J$ with the helicity operator which in the present frame is $J_3 = M_{12}$ one sees that Q_2^I lowers the helicity by one half and $\overline{Q}_2^J$ rises it by one half. (For simplicity, we suppose here that the state $|\lambda_0\rangle$ transforms as a singlet under the SU(N) that acts on the indices I, J. One could easily drop this restriction.) The supermultiplet then is of the form

$$\begin{aligned} & |\lambda_0\rangle \\ a_I^\dagger |\lambda_0\rangle &= \left|\lambda_0 + \tfrac{1}{2}\right\rangle_I \\ a_I^\dagger a_J^\dagger |\lambda_0\rangle &= |\lambda_0 + 1\rangle_{IJ} \\ & \ldots \\ a_1^\dagger a_2^\dagger \ldots a_N^\dagger |\lambda_0\rangle &= \left|\lambda_0 + \tfrac{N}{2}\right\rangle . \end{aligned} \tag{3.11}$$

Due to the antisymmetry in $I, J, \ldots$ there are $\binom{N}{k}$ states with helicity $\lambda = \lambda_0 + \frac{k}{2}$, $k = 0, 1, \ldots N$. Summing the binomial coefficients gives a total of 2^N states with 2^{N-1} having integer helicity (bosons) and 2^{N-1} having half-integer helicity (fermions). In general, in such a supermultiplet, except if $\lambda_0 = -\frac{N}{4}$, the helicities will not be distributed symmetrically about zero. Such supermultiplets cannot be invariant under CPT, since CPT flips the sign of the helicity. To satisfy CPT one then need to double these multiplets by adding their CPT conjugate with opposite helicities and opposite quantum numbers.

For unextended susy, $N = 1$, each massless supermultiplet only contains two states $|\lambda\rangle_0$ and $\left|\lambda_0 + \frac{1}{2}\right\rangle$. We denote these multiplets by $(\lambda_0, \lambda_0 + \frac{1}{2})$. They can never be CPT self-conjugate and one needs to double them. Thus one arrives at the following massless $N = 1$ multiplets:

<u>The chiral multiplet</u> consists of $(0, \frac{1}{2})$ and its CPT conjugate $(-\frac{1}{2}, 0)$, corresponding to a Weyl fermion and a complex scalar.

<u>The vector multiplet</u> consists of $(\frac{1}{2}, 1)$ plus $(-1, -\frac{1}{2})$, corresponding to a gauge boson (massless vector) and a Weyl fermion, both necessarily in the adjoint representation of the gauge group.

<u>The gravitino multiplet</u> contains $(1, \frac{3}{2})$ and $(-\frac{3}{2}, -1)$, i.e. a gravitino and a gauge boson.

<u>The graviton multiplet</u> contains $(\frac{3}{2}, 2)$ and $(-2, -\frac{3}{2})$, corresponding to the graviton and the gravitino.

Since we so not want helicities larger than two, we must stop here. Also the gravitino should be present only in a theory with gravity, so if $N=1$ it must only occur once and then in the gravity multiplet. Hence the gravitino multiplet cannot appear in unextended susy. However it does appear in extended susy when decomposing larger multiplets into $N=1$ multiplets.

For $N=2$ a supermultiplet contains $(\lambda_0, \lambda_0+\frac{1}{2}, \lambda_0+\frac{1}{2}, \lambda_0+1)$. Restricting ourselves to the cases where the helicity does not exceed one, we have two possibilities.

The $N=2$ vector multiplet contains $(0, \frac{1}{2}, \frac{1}{2}, 1)$ and its CPT conjugate $(-1, -\frac{1}{2}, -\frac{1}{2}, 0)$, corresponding to a vector (gauge boson), two Weyl fermions and a complex scalar, again all in the adjoint representation of the gauge group. In terms of $N=1$ representations this is a vector and a chiral $N=1$ multiplet.

The hypermultiplet: If $\lambda_0 = -\frac{1}{2}$ we get $(-\frac{1}{2}, 0, 0\frac{1}{2})$. This may or not be CPT self-conjugate. If it is, it is called a half-hypermultiplet. If it is not we have to add its CPT conjugate to get a (full) hypermultiplet $2\times(-\frac{1}{2}, 0, 0\frac{1}{2})$.

For $N=4$, restricting again to helicities not exceeding one, there is a single $N=4$ multiplet which always is CPT self-conjugate. It is $(-1, 4\times(-\frac{1}{2}), 6\times 0, 4\times\frac{1}{2}, 1)$, containing a vector 4 Weyl fermions and 3 complex scalars. In terms of $N=2$ multiplets it is just the sum of the $N=2$ vector multiplet and a hypermultiplet, however now all transforming in the adjoint of the gauge group.

3.4 Massive supermultiplets

We now consider the case $P^2>0$ and a priori arbitrary central charges Z^{IJ}. Going to the rest frame $P_\mu=(m,0,0,0)$, the susy algebra becomes

$$\begin{aligned}
\{Q^I_\alpha, (Q^J_\beta)^\dagger\} &= 2m\delta_{\alpha\beta}\delta^{IJ} \\
\{Q^I_\alpha, Q^J_\beta\} &= \epsilon_{\alpha\beta}Z^{IJ} \\
\{(Q^I_\alpha)^\dagger, (Q^J_\beta)^\dagger\} &= \epsilon_{\alpha\beta}(Z^{IJ})^* \ .
\end{aligned} \tag{3.12}$$

By an appropriate U(N) rotation among the Q^I the antisymmetric matrix of central charges can be brought into standard form:

$$Z^{IJ} = \begin{pmatrix} 0 & q_1 & 0 & 0 & \\ -q_1 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & q_2 & \\ 0 & 0 & -q_2 & 0 & \\ & & \vdots & & \end{pmatrix} \tag{3.13}$$

with all $q_n \geq 0$, $n = 1, \ldots \frac{N}{2}$. We assume that N is even, since otherwise there is an extra zero eigenvalue of the Z-matrix which can be handled trivially.

It follows that if we let

$$\begin{aligned} a^1_\alpha &= \tfrac{1}{\sqrt{2}}\left(Q^1_\alpha + \epsilon_{\alpha\beta}(Q^2_\beta)^\dagger\right) \\ b^1_\alpha &= \tfrac{1}{\sqrt{2}}\left(Q^1_\alpha - \epsilon_{\alpha\beta}(Q^2_\beta)^\dagger\right) \\ a^2_\alpha &= \tfrac{1}{\sqrt{2}}\left(Q^3_\alpha + \epsilon_{\alpha\beta}(Q^4_\beta)^\dagger\right) \\ b^2_\alpha &= \tfrac{1}{\sqrt{2}}\left(Q^3_\alpha - \epsilon_{\alpha\beta}(Q^4_\beta)^\dagger\right) \\ &\vdots \end{aligned} \tag{3.14}$$

then the a^r_α and b^r_α, $r = 1, \ldots \frac{N}{2}$ and their hermitian conjugates satisfy the following algebra of harmonic oscillators

$$\begin{aligned} \{a^r_\alpha, (a^s_\beta)^\dagger\} &= (2m - q_r)\delta_{rs}\delta_{\alpha\beta} \\ \{b^r_\alpha, (b^s_\beta)^\dagger\} &= (2m + q_r)\delta_{rs}\delta_{\alpha\beta} \\ \{a^r_\alpha, (b^s_\beta)^\dagger\} &= \{a^r_\alpha, a^s_\beta\} = \ldots = 0 \end{aligned} \tag{3.15}$$

Clearly, positivity of the Hilbert space requires that

$$2m \geq |q_n| \tag{3.16}$$

for all n. If some of the q_n saturate the bound, i.e. are equal to m, then the corresponding operators must be set to zero, as we did in the massless case with the Q^I_1. Clearly, in the massless case the bound becomes $0 \geq |q_n|$ and thus $q_n = 0$ always. There cannot be central charges in the massless case and the bound is always saturated, thus only exactly have of the fermionic generators survive.

In the more general massive case, if all $|q_n|$ are strictly less than $2m$ we have a total of $2N$ harmonic oscillators. Then starting from a state of minimal "helicity" (i.e. z component of the angular momentum) λ_0 annihilated by all a^n_α and b^n_β, application of the creation operators yields a total of 2^{2N} states with helicities ranging from λ_0 to $\lambda_0 + N$.

For $N = 1$ this yields four states, again labeled by their helicities (or rather the z component of the angular momentum), as $(-\frac{1}{2}, 0, 0, \frac{1}{2})$ (which is the same as the CPT extended massless multiplet) or $(-1, -\frac{1}{2}, -\frac{1}{2}, 0)$ to which we must add the CPT conjugate $(1, \frac{1}{2}, \frac{1}{2}, 0)$. The latter are the same states as a massless vector plus a massless chiral multiplet and can be obtained from them via a Higgs mechanism. In terms of massive representations this is a vector (3 dofs) a Dirac fermion (4 dofs) and a single real scalar (1 dof).

For $N = 2$ we already have 16 states with helicities ranging at least from -1 to 1. Such a massive $N = 2$ multiplet can be viewed as the union of a massless $N = 2$ vector and hypermultiplet. A generic massive $N = 4$ multiplet contains $2^8 = 256$ states including at least a helicity ± 2. Thus such a theory must include a massive spin two particle which is not believed to be possible in quantum field theory.

If $k < \frac{N}{2}$ of the q_n are equal to $2m$ then we only have $2N - 2k$ oscillators, and the supermultiplets will only contain $2^{2(N-k)}$ states. They are called short multiplets or BPS multiplets. If all q_n equal $2m$, i.e. $k = \frac{N}{2}$ we get the shortest multiplets with only 2^N states, exactly as in the massless case. These BPS multiplets are also called ultrashort, and are completely analogous to the massless multiplets.

Chapter 4

Superspace and superfields

Since we want to construct supersymmetric quantum field theories, we have to find representations of the susy algebra on fields. A convenient and compact way to do this is to introduce superspace and superfields, i.e. fields defined on superspace. This is particularly simple for unextended susy, so we will restrict here to $N = 1$ superspace and superfields. Then we have two plus two susy generators Q_α and $\overline{Q}_{\dot{\alpha}}$, as well as four generators P_μ of space-time translations. The idea then is to enlarge space-time labelled by the coordinates x^μ by adding two plus two anticommuting Grassmannian coordinates θ_α and $\overline{\theta}_{\dot{\alpha}}$. Thus coordinates on superspace are $(x^\mu, \theta_\alpha, \overline{\theta}_{\dot{\alpha}})$. Rather than elaborating on the meaning of such a space we will simply use it as a very efficient recepee to perform calculations in supersymmetric theories.

4.1 Superspace

As already said, we restrict here to $N = 1$. The "odd" superspace coordinates θ_α and $\overline{\theta}_{\dot{\alpha}}$ just behave as constant (x^μ independent) spinors. Recall that as all spinors they anticommute among themselves, i.e. $\theta^1\theta^2 = -\theta^2\theta^1$, and idem for the $\overline{\theta}^{\dot{\alpha}}$. Spinor indices in bilinears are contracted acording to the usual rule, i.e. $\theta\theta = \theta^\alpha\theta_\alpha = -2\theta^1\theta^2 = +2\theta_2\theta_1 = -2\theta_1\theta_2$, and $\overline{\theta\theta} = \overline{\theta}_{\dot{\alpha}}\overline{\theta}^{\dot{\alpha}} = 2\overline{\theta}_1\overline{\theta}_2 = \ldots$. One can then easily prove the following useful identities:

$$\begin{aligned} \theta^\alpha\theta^\beta = -\tfrac{1}{2}\epsilon^{\alpha\beta}\theta\theta \quad &, \quad \overline{\theta}^{\dot{\alpha}}\overline{\theta}^{\dot{\beta}} = \tfrac{1}{2}\epsilon^{\dot{\alpha}\dot{\beta}}\overline{\theta\theta}\ , \\ \theta_\alpha\theta_\beta = \tfrac{1}{2}\epsilon_{\alpha\beta}\theta\theta \quad &, \quad \overline{\theta}_{\dot{\alpha}}\overline{\theta}_{\dot{\beta}} = -\tfrac{1}{2}\epsilon_{\dot{\alpha}\dot{\beta}}\overline{\theta\theta}\ , \\ \theta\sigma^\mu\overline{\theta}\,\theta\sigma^\nu\overline{\theta} = \tfrac{1}{2}\theta\theta\,\overline{\theta\theta}g^{\mu\nu} \quad &, \quad \theta\psi\ \theta\chi = -\tfrac{1}{2}\theta\theta\ \psi\chi\ . \end{aligned} \tag{4.1}$$

Derivatives in θ and $\overline{\theta}$ are defined in an obvious way as $\frac{\partial}{\theta^\alpha}\theta^\beta = \delta_\alpha^\beta$ and $\frac{\partial}{\overline{\theta}^{\dot\alpha}}\overline{\theta}^{\dot\beta} = \delta_{\dot\alpha}^{\dot\beta}$. Since the θ's anticommute, any product involving more than two θ's or more than two $\overline{\theta}$'s vanishes. Hence an arbitrary (scalar) function on superspace, i.e. a superfield, can always be expanded as

$$\begin{aligned} F(x,\theta,\overline{\theta}) &= f(x) + \theta\psi(x) + \overline{\theta}\overline{\chi}(x) + \theta\theta\, m(x) + \overline{\theta\theta}\, n(x) \\ &+ \theta\sigma^\mu\overline{\theta}\, v_\mu(x) + \theta\theta\,\overline{\theta}\overline{\lambda}(x) + \overline{\theta\theta}\,\theta\rho(x) + \theta\theta\,\overline{\theta\theta}\, d(x)\ . \end{aligned} \tag{4.2}$$

If F carries extra vector indices then so do the fomponent fields $f, \psi, \ldots$.

Integration on superspace is defined for a single Grassmannian variable, say θ^1 as $\int \mathrm{d}\theta^1(a+\theta^1 b) = b$ so that $\int \mathrm{d}\theta^1\mathrm{d}\theta^2\theta^2\theta^1 = 1$. Then since $\theta\theta = 2\theta^2\theta^1$ and $\overline{\theta\theta} = 2\theta^1\theta^2$ we define $\mathrm{d}^2\theta = \frac{1}{2}\mathrm{d}\theta^1\mathrm{d}\theta^2$ and $\mathrm{d}^2\overline{\theta} = \frac{1}{2}\mathrm{d}\overline{\theta}^2\mathrm{d}\overline{\theta}^1 = [\mathrm{d}^2\theta]^\dagger$ so that

$$\int \mathrm{d}^2\theta\ \theta\theta = \int \mathrm{d}^2\overline{\theta}\ \overline{\theta\theta} = 1\ . \tag{4.3}$$

It is easy to check that

$$\int \mathrm{d}^2\theta\ = \frac{1}{4}\epsilon^{\alpha\beta}\frac{\partial}{\partial\theta^\alpha}\frac{\partial}{\partial\theta^\beta}\quad ,\quad \int \mathrm{d}^2\overline{\theta}\ = -\frac{1}{4}\epsilon^{\dot\alpha\dot\beta}\frac{\partial}{\partial\overline{\theta}^{\dot\alpha}}\frac{\partial}{\partial\overline{\theta}^{\dot\beta}}\ . \tag{4.4}$$

Clearly one also has

$$\int \mathrm{d}^2\theta\mathrm{d}^2\overline{\theta}\ \theta\theta\overline{\theta\theta} = 1\ . \tag{4.5}$$

With these definitions it is easy to see that one has the hermiticity property

$$\left(\frac{\partial}{\theta^\alpha}\right)^\dagger = +\frac{\partial}{\overline{\theta}^{\dot\alpha}} \tag{4.6}$$

with $\alpha \equiv \dot\alpha$. Note the plus sign rather than a minus sign as one would expect from $(\partial_\mu)^\dagger = -\partial_\mu$.

We now want to realise the susy generators Q_α and their hermitian conjugates $\overline{Q}_{\dot\alpha} = (Q_\alpha)^\dagger$ as differential operators on superspace. We want that $i\epsilon^\alpha Q_\alpha$ generates a translation in θ^α by a constant infinitesimal spinor ϵ^α plus some translation in x^μ. The latter space-time translation is determined by the susy algebra since the commutator of two such susy transformations is a translation in space-time. Thus we want

$$(1+i\epsilon Q)F(x,\theta,\theta) = F(x+\delta x, \theta+\epsilon, \overline{\theta})\ . \tag{4.7}$$

Hence $iQ_\alpha = \frac{\partial}{\partial\theta^\alpha} + \ldots$ where $+\ldots$ must be of the form $c(\sigma^\mu\overline{\theta})_\alpha P_\mu = -ic(\sigma^\mu\overline{\theta})_\alpha\partial_\mu$ with some constant c to be determined. We arrive at the ansatz

$$Q_\alpha = -i\left(\frac{\partial}{\partial\theta^\alpha} - ic(\sigma^\mu\overline{\theta})_\alpha\partial_\mu\right)\ . \tag{4.8}$$

Then the hermitian conjugate is

$$\overline{Q}_{\dot{\alpha}} = i\left(\frac{\partial}{\partial\overline{\theta}^{\dot{\alpha}}} - ic^*(\theta\sigma^\mu)_{\dot{\alpha}}\partial_\mu\right) \ , \tag{4.9}$$

and they satisfy the susy algebra, in particular

$$\{Q_\alpha, Q_{\dot{\beta}}\} = 2\sigma^\mu_{\alpha\dot{\beta}}P_\mu = -2i\sigma^\mu_{\alpha\dot{\beta}}\partial_\mu \tag{4.10}$$

if Re $c = 1$. We choose $c = 1$ so that

$$\begin{aligned} Q_\alpha &= -i\tfrac{\partial}{\partial\theta^\alpha} - \sigma^\mu_{\alpha\dot{\beta}}\overline{\theta}^{\dot{\beta}}\partial_\mu \\ \overline{Q}_{\dot{\alpha}} &= i\tfrac{\partial}{\partial\overline{\theta}^{\dot{\alpha}}} + \theta^\beta\sigma^\mu_{\beta\dot{\alpha}}\partial_\mu \ . \end{aligned} \tag{4.11}$$

We can now give the action on the superfield F and determine δx:

$$(1 + i\epsilon Q + i\overline{\epsilon}\overline{Q})F(x^\mu, \theta^\alpha, \overline{\theta}^{\dot{\beta}}) = F(x^\mu - i\epsilon\sigma^\mu\overline{\theta} + i\theta\sigma^\mu\overline{\epsilon}, \theta^\alpha + \epsilon^\alpha, \overline{\theta}^{\dot{\beta}} + \overline{\epsilon}^{\dot{\beta}}) \tag{4.12}$$

and the susy variation of a superfield is of course defined as

$$\delta_{\epsilon,\overline{\epsilon}}F = (i\epsilon Q + i\overline{\epsilon}\overline{Q})F \ . \tag{4.13}$$

Since a general superfield contains too many component fields to correspond to an irreducible representation of $N = 1$ susy, it will be very useful to impose susy invariant condition to lower the number of components. To do this, we first find covariant derivatives D_α and $\overline{D}_{\dot{\alpha}}$ that anticommute with the susy generators Q and $\overline{Q}$. Then $\delta_{\epsilon,\overline{\epsilon}}(D_\alpha F) = D_\alpha(\delta_{\epsilon,\overline{\epsilon}}F)$ and idem for $\overline{D}_{\dot{\alpha}}$. It follows that $D_\alpha F = 0$ or $\overline{D}_{\dot{\alpha}}F = 0$ are susy invariant constraints one may impose to reduce the number of components in a superfield. One finds

$$\begin{aligned} D_\alpha &= \tfrac{\partial}{\partial\theta^\alpha} + i\sigma^\mu_{\alpha\dot{\beta}}\overline{\theta}^{\dot{\beta}}\partial_\mu \\ \overline{D}_{\dot{\alpha}} &= = \tfrac{\partial}{\partial\overline{\theta}^{\dot{\alpha}}} + i\theta^\beta\sigma^\mu_{\beta\dot{\alpha}}\partial_\mu \end{aligned} \tag{4.14}$$

where $\overline{D}_{\dot{\alpha}} = (D_\alpha)^\dagger$ and

$$\begin{aligned} \{D_\alpha, \overline{D}_{\dot{\beta}}\} = 2i\sigma^\mu_{\alpha\dot{\beta}}\partial_\mu \quad &, \quad \{D_\alpha, D_\beta\} = \{\overline{D}_{\dot{\alpha}}, \overline{D}_{\dot{\beta}}\} = 0 \\ \{D_\alpha, Q_\beta\} = \{\overline{D}_{\dot{\alpha}}, Q_\beta\} \ &= \ \{D_\alpha, \overline{Q}_{\dot{\beta}}\} = \{\overline{D}_{\dot{\alpha}}, \overline{Q}_{\dot{\beta}}\} = 0 \ . \end{aligned} \tag{4.15}$$

4.2 Chiral superfields

A chiral superfield ϕ is defined by the condition

$$\overline{D}_{\dot\alpha}\phi = 0 \tag{4.16}$$

and an anti-chiral one $\overline{\phi}$ by

$$D_\alpha\overline{\phi} = 0\ . \tag{4.17}$$

This is easily solved by observing that

$$\begin{aligned} D_\alpha\overline{\theta} = \overline{D}_{\dot\alpha}\theta &= D_\alpha\overline{y}^\mu = \overline{D}_{\dot\alpha}y^\mu = 0\ , \\ y^\mu = x^\mu + i\theta\sigma^\mu\overline{\theta} \qquad &, \qquad \overline{y}^\mu = x^\mu - i\theta\sigma^\mu\overline{\theta}\ . \end{aligned} \tag{4.18}$$

Hence ϕ depends only on θ and y^μ (i.e. all $\overline{\theta}$ dependence is through y^μ) and $\overline{\phi}$ only on $\overline{\theta}$ and $\overline{y}^\mu$. Concentrating on ϕ we have the component expansion

$$\phi(y,\theta) = z(y) + \sqrt{2}\theta\psi(y) - \theta\theta f(y) \tag{4.19}$$

or Taylor expanding in terms of x, θ and $\overline{\theta}$:

$$\begin{aligned} \phi(y,\theta) &= z(x) + \sqrt{2}\theta\psi(x) + i\theta\sigma^\mu\overline{\theta}\partial_\mu z(x) - \theta\theta f(x) \\ &- \tfrac{i}{\sqrt{2}}\theta\theta\partial_\mu\psi(x)\sigma^\mu\overline{\theta} - \tfrac{1}{4}\theta\theta\overline{\theta\theta}\partial^2 z(x)\ . \end{aligned} \tag{4.20}$$

Physically, such a chiral superfield describes one complex scalar z and one Weyl fermion ψ. The field f will turn out to be an auxiliary field. For $\overline{\phi}$ we similarly have

$$\begin{aligned} \overline{\phi}(y,\overline{\theta}) &= \overline{z}(\overline{y}) + \sqrt{2}\overline{\theta\psi}(\overline{y}) - \overline{\theta\theta f}(\overline{y}) \\ &= \overline{z}(x) + \sqrt{2}\overline{\theta\psi}(x) - i\theta\sigma^\mu\overline{\theta}\partial_\mu\overline{z}(x) - \overline{\theta\theta f}(x) \\ &+ \tfrac{i}{\sqrt{2}}\overline{\theta\theta}\theta\sigma^\mu\partial_\mu\overline{\psi}(x) - \tfrac{1}{4}\theta\theta\overline{\theta\theta}\partial^2\overline{z}(x)\ . \end{aligned} \tag{4.21}$$

Finally, let us find the explicit susy variations of the component fields as it results from (4.13): First, for chiral superfields it is useful to change variables from $x^\mu, \theta, \overline{\theta}$ to $y^\mu, \theta, \overline{\theta}$. Then

$$Q_\alpha = -i\frac{\partial}{\partial\theta^\alpha} \quad , \quad \overline{Q}_{\dot\alpha} = i\frac{\partial}{\partial\overline{\theta}^{\dot\alpha}} + 2\theta^\beta\sigma^\mu_{\beta\dot\alpha}\frac{\partial}{\partial y^\mu} \tag{4.22}$$

so that

$$\begin{aligned} \delta\phi(y,\theta) &\equiv \left(i\epsilon Q + i\overline{\epsilon Q}\right)\phi(y,\theta) = \left(\epsilon^\alpha\tfrac{\partial}{\partial\theta^\alpha} + 2i\theta\sigma^\mu\overline{\epsilon}\tfrac{\partial}{\partial y^\mu}\right)\phi(y,\theta) \\ &= \sqrt{2}\epsilon\psi - 2\epsilon\theta f + 2i\theta\sigma^\mu\overline{\epsilon}(\partial_\mu z + \sqrt{2}\theta\partial_\mu\psi) \\ &= \sqrt{2}\epsilon\psi + \sqrt{2}\theta\left(-\sqrt{2}\epsilon f + \sqrt{2}i\sigma^\mu\overline{\epsilon}\partial_\mu z\right) - \theta\theta\left(-i\sqrt{2}\overline{\epsilon\sigma}^\mu\partial_\mu\psi\right)\ . \end{aligned} \tag{4.23}$$

Thus we read the susy transformations of the component fields:

$$\begin{aligned}
\delta z &= \sqrt{2}\epsilon\psi \\
\delta\psi &= \sqrt{2}i\partial_\mu z\sigma^\mu\overline{\epsilon} - \sqrt{2}f\epsilon \\
\delta f &= \sqrt{2}i\partial_\mu\psi\sigma^\mu\overline{\epsilon}\ .
\end{aligned} \tag{4.24}$$

The factors of $\sqrt{2}$ do appear because of our normalisations of the fields and the definition of $\delta\phi$. If desired, they could be absorbed by a rescaling of ϵ and $\overline{\epsilon}$.

4.3 Susy invariant actions

To construct susy invariant actions we now only need to make a few observations. First, products of superfields are of course superfields. Also, products of (anti) chiral superfields are still (anti) chiral superfields. Typically, one will have a superpotential $W(\phi)$ which is again chiral. This W may depend on several different ϕ_i. Using the y and θ variables one easily Taylor expands

$$\begin{aligned}
W(\phi) &= W(z(y)) + \sqrt{2}\tfrac{\partial W}{\partial z_i}\,\theta\psi_i(y) \\
&- \theta\theta\left(\tfrac{\partial W}{\partial z_i}\,f_i(y) + \tfrac{1}{2}\tfrac{\partial^2 W}{\partial z_i\partial z_j}\,\psi_i(y)\psi_j(y)\right)
\end{aligned} \tag{4.25}$$

where it is understood that $\partial W/\partial z$ and $\partial^2 W/\partial z\partial z$ are evaluated at $z(y)$. The second and important observation is that any Lagrangian of the form

$$\int \mathrm{d}^2\theta\mathrm{d}^2\overline{\theta}\ F(x,\theta,\overline{\theta}) + \int \mathrm{d}^2\theta\ W(\phi) + \int \mathrm{d}^2\overline{\theta}\ [W(\phi)]^\dagger \tag{4.26}$$

is automatically susy invariant, i.e. it transforms at most by a total derivative in space-time. The proof is very simple. The susy variation of any superfield is given by (4.13) and, since the ϵ and $\overline{\epsilon}$ are constant spinors and the Q and $\overline{Q}$ are differential operators in superspace, it is again a total derivative in all of superspace:

$$\delta F = \frac{\partial}{\partial\theta^\alpha}(-\epsilon^\alpha F) + \frac{\partial}{\partial\overline{\theta}^{\dot\alpha}}(-\overline{\epsilon}^{\dot\alpha} F) + \frac{\partial}{\partial x^\mu}[-i(\epsilon\sigma^\mu\overline{\theta} - \theta\sigma^\mu\overline{\epsilon})F]\ . \tag{4.27}$$

Integration $\int \mathrm{d}^2\theta\mathrm{d}^2\overline{\theta}$ only leaves the last term which is a total space-time derivative as claimed. If now F is a chiral superfield like ϕ or $W(\phi)$ one changes variables to θ and y and one has

$$\delta\phi = \frac{\partial}{\partial\theta^\alpha}(-\epsilon^\alpha\phi(y,\theta)) + \frac{\partial}{\partial y^\mu}[-i(\epsilon\sigma^\mu\overline{\theta} - \theta\sigma^\mu\overline{\epsilon})\phi(y,\theta)]\ . \tag{4.28}$$

Integrating $\int \mathrm{d}^2\theta$ again only leaves the last term which becomes $\frac{\partial}{\partial x^\mu}[\ldots]$ and is a total derivative in space-time. The analogous result holds for an anti chiral superfield $\overline{W}(\overline{\phi}) = [W(\phi)]^\dagger$ and integration $\int \mathrm{d}^2\overline{\theta}$. This proves the supersymmetry of the action resulting from the space-time integral of the Lagrangian (4.26).

The terms $\int \mathrm{d}^2\theta\ W(\phi) + h.c.$ in the Lagrangian have the form of a potential. The kinetic terms must be provided by the term $\int \mathrm{d}^2\theta \mathrm{d}^2\overline{\theta}\ F$. The simplest choice is $F = \phi^\dagger\phi$. This is neither chiral nor anti chiral but real. To compute $\phi^\dagger\phi$ one must first expand the y^μ in terms of x^μ. We only need the terms $\sim \theta\theta\overline{\theta}\overline{\theta}$, called the D-term:

$$\begin{aligned}\phi^\dagger\phi\Big|_{\theta\theta\overline{\theta}\overline{\theta}} &= -\tfrac{1}{4}z^\dagger\partial^2 z - \tfrac{1}{4}\partial^2 z^\dagger z + \tfrac{1}{2}\partial_\mu z^\dagger\partial^\mu z + f^\dagger f + \tfrac{i}{2}\partial_\mu\psi\sigma^\mu\overline{\psi} - \tfrac{i}{2}\psi\sigma^\mu\partial_\mu\overline{\psi} \\ &= \partial_\mu z^\dagger\partial^\mu z + \tfrac{i}{2}(\partial_\mu\psi\sigma^\mu\overline{\psi} - \psi\sigma^\mu\partial_\mu\overline{\psi}) + f^\dagger f + \text{total derivative}\ .\end{aligned} \tag{4.29}$$

Then

$$S = \int \mathrm{d}^4x\mathrm{d}^2\theta\mathrm{d}^2\overline{\theta}\ \ \overline{\phi}_i^\dagger\phi_i + \int \mathrm{d}^4x\mathrm{d}^2\theta\ W(\phi_i) + h.c. \tag{4.30}$$

yields

$$S = \int \mathrm{d}^4x\Big[|\partial_\mu z_i|^2 - i\psi_i\sigma^\mu\partial_\mu\overline{\psi}_i + f_i^\dagger f_i - \frac{\partial W}{\partial z_i}f_i + h.c. - \frac{1}{2}\frac{\partial^2 W}{\partial z_i\partial z_j}\psi_i\psi_j + h.c.\Big]\ . \tag{4.31}$$

More generally, one can replace $\phi_i^\dagger\phi_i$ by a (real) Kähler potential $K(\phi_i^\dagger, \phi_j)$. This leads to the non-linear σ-model discussed later. In any case, the f_i have no kinetic term and hence are auxiliary fields. They should be eliminated by substituting their algebraic equations of motion

$$f_i^\dagger = \left(\frac{\partial W}{\partial z_i}\right) \tag{4.32}$$

into the action, leading to

$$S = \int \mathrm{d}^4x\Big[|\partial_\mu z_i|^2 - i\psi_i\sigma^\mu\partial_\mu\overline{\psi}_i - \left|\frac{\partial W}{\partial z_i}\right|^2 - \frac{1}{2}\frac{\partial^2 W}{\partial z_i\partial z_j}\psi_i\psi_j - \frac{1}{2}\left(\frac{\partial^2 W}{\partial z_i\partial z_j}\right)^\dagger\overline{\psi}_i\overline{\psi}_j\Big]\ . \tag{4.33}$$

We see that the scalar potential V is determined in terms of the superpotential W as

$$V = \sum_i \left|\frac{\partial W}{\partial z_i}\right|^2\ . \tag{4.34}$$

To illustrate this model, consider the simplest case of a single chiral superfield ϕ and a cubic superpotential $W(\phi) = \frac{m}{2}\phi^2 + \frac{g}{3}\phi^3$. Then $\frac{\partial W}{\partial z} = m\phi + g\phi^2$ and the action becomes

$$S_{\rm WZ} = \int {\rm d}^4x \quad \Big[\quad |\partial_\mu z|^2 - i\psi\sigma^\mu\partial_\mu\overline{\psi} - m^2|z|^2 - \tfrac{m}{2}(\psi\psi + \overline{\psi\psi}) \\ - \quad mg(z^\dagger z^2 + (z^\dagger)^2 z) - g^2|z|^4 + g(z\psi\psi + z^\dagger\overline{\psi\psi})\Big] \ . \tag{4.35}$$

Note that the Yukawa interactions appear with a coupling constant g that is related by susy to the bosonic coupling constants mg and g^2.

4.4 Vector superfields

The $N = 1$ supermultiplet of next higher spin is the vector multiplet. The corresponding superfield $V(x, \theta, \overline{\theta})$ is real and has the expansion

$$\begin{aligned} V(x,\theta,\overline{\theta}) &= C + i\theta\chi - i\overline{\theta}\overline{\chi} + \theta\sigma^\mu\overline{\theta}v_\mu \\ &+ \tfrac{i}{2}\theta\theta(M+iN) - \tfrac{i}{2}\overline{\theta\theta}(M-iN) \\ &+ i\theta\theta\,\overline{\theta}\left(\overline{\lambda} + \tfrac{i}{2}\overline{\sigma}^\mu\partial_\mu\chi\right) - i\overline{\theta\theta}\,\theta\left(\lambda - \tfrac{i}{2}\sigma^\mu\partial_\mu\overline{\chi}\right) \\ &+ \tfrac{1}{2}\theta\theta\overline{\theta\theta}\left(D - \tfrac{1}{2}\partial^2 C\right) \end{aligned} \tag{4.36}$$

where all component fields only depend on x^μ. There are 8 bosonic components (C, D, M, N, v_μ) and 8 fermionic components (χ, λ). These are too many components to describe a single supermultiplet. We want to reduce their number by making use of the supersymmetric generalisation of a gauge transformation. Note that the transformation

$$V \to V + \phi + \phi^\dagger \ , \tag{4.37}$$

with ϕ a chiral superfield, implies the component transformation

$$v_\mu \to v_\mu + \partial_\mu(2{\rm Im}z) \tag{4.38}$$

which is an abelian gauge transformation. We conclude that (4.37) is its desired supersymmetric generalisation. If this transformation (4.37) is a symmetry (actually a gauge symmetry, as we just saw) of the theory then, by an appropriate choice of ϕ, one can transform away the components χ, C, M, N and one component of v_μ. This choice is called the Wess-Zumino gauge, and it reduces the vector superfield to

$$V_{\rm WZ} = \theta\sigma^\mu\overline{\theta}v_\mu(x) + i\theta\theta\,\overline{\theta\lambda}(x) - i\overline{\theta\theta}\,\theta\lambda(x) + \frac{1}{2}\theta\theta\overline{\theta\theta}D(x) \ . \tag{4.39}$$

Since each term contains at least one θ, the only non-vanishing power of $V_{\rm WZ}$ is

$$V_{\rm WZ}^2 = \theta\sigma^\mu\overline{\theta}\ \theta\sigma^\nu\overline{\theta}\ v_\mu v_\nu = \frac{1}{2}\theta\theta\overline{\theta\theta}\ v_\mu v^\mu \tag{4.40}$$

and $V_{\rm WZ}^n = 0$, $n \geq 3$.

To construct kinetic terms for the vector field v_μ one must act on V with the covariant derivatives D and $\overline{D}$. Define

$$W_\alpha = -\frac{1}{4}\overline{DD}D_\alpha V \quad , \quad \overline{W}_{\dot\alpha} = -\frac{1}{4}DD\overline{D}_{\dot\alpha}V \ . \tag{4.41}$$

(This is appropriate for abelian gauge theories and will be slightly generalized in the non-abelian case.) Since $D^3 = \overline{D}^3 = 0$, W_α is chiral and $\overline{W}_{\dot\alpha}$ antichiral. Furthermore it is clear that they behave as anticommuting Lorentz spinors. Note that they are invariant under the transformation (4.37) since

$$\begin{aligned} W_\alpha \quad &\rightarrow \quad W_\alpha - \tfrac{1}{4}\overline{DD}D_\alpha(\phi+\phi^\dagger) = W_\alpha + \tfrac{1}{4}\overline{D}^{\dot\beta}\overline{D}_{\dot\beta}D_\alpha\phi \\ &= \quad W_\alpha + \tfrac{1}{4}\overline{D}^{\dot\beta}\{\overline{D}_{\dot\beta}, D_\alpha\}\phi = W_\alpha + \tfrac{i}{2}\sigma^\mu_{\alpha\dot\beta}\partial_\mu\overline{D}^{\dot\beta}\phi = W_\alpha \end{aligned} \tag{4.42}$$

since $\overline{D}\phi = D\phi^\dagger = 0$. It is then easiest to use the WZ-gauge to compute W_α. To facilitate things further, change variables to $y^\mu, \theta^\alpha, \overline{\theta}^{\dot\alpha}$ so that

$$D_\alpha = \frac{\partial}{\partial\theta^\alpha} + 2i\sigma^\mu_{\alpha\dot\beta}\overline{\theta}^{\dot\beta}\frac{\partial}{\partial y^\mu} \quad , \quad \overline{D}_{\dot\alpha} = \frac{\partial}{\partial\overline{\theta}^{\dot\alpha}} \tag{4.43}$$

and write

$$V_{\rm WZ} = \theta\sigma^\mu\overline{\theta}v_\mu(y) + i\theta\theta\,\overline{\theta\lambda}(y) - i\overline{\theta\theta}\,\theta\lambda(y) + \frac{1}{2}\theta\theta\overline{\theta\theta}\,(D(y) - i\partial_\mu v^\mu(y)) \ . \tag{4.44}$$

Then, using $\sigma^\nu\overline{\sigma}^\mu - g^{\nu\mu} = 2\sigma^{\nu\mu}$, it is straightforward to find (all arguments are y^μ)

$$\begin{aligned} D_\alpha V_{\rm WZ} \quad &= \quad (\sigma^\mu\overline{\theta})_\alpha v_\mu + 2i\theta_\alpha\,\overline{\theta\lambda} - i\overline{\theta\theta}\,\lambda_\alpha + \theta_\alpha\,\overline{\theta\theta}D \\ &+ \quad 2i(\sigma^{\mu\nu}\theta)_\alpha\overline{\theta\theta}\partial_\mu v_\nu + \theta\theta\overline{\theta\theta}(\sigma^\mu\partial_\mu\overline{\lambda})_\alpha \end{aligned} \tag{4.45}$$

and then, using $\overline{DD\theta\theta} = -4$,

$$W_\alpha = -i\lambda_\alpha(y) + \theta_\alpha D(y) + i(\sigma^{\mu\nu}\theta)_\alpha f_{\mu\nu}(y) + \theta\theta(\sigma^\mu\partial_\mu\overline{\lambda}(y))_\alpha \tag{4.46}$$

with

$$f_{\mu\nu} = \partial_\mu v_\nu - \partial_\nu v_\mu \tag{4.47}$$

being the abelian field strength associated with v_μ.

Since W_α is a chiral superfield, $\int d^2\theta\ W^\alpha W_\alpha$ will be a susy invariant Lagrangian. To obtain its component expansion we need the $\theta\theta$-term (F-term) of $W^\alpha W_\alpha$:

$$W^\alpha W_\alpha\Big|_{\theta\theta} = -2i\lambda\sigma^\mu\partial_\mu\overline{\lambda} + D^2 - \frac{1}{2}(\sigma^{\mu\nu})^{\alpha\beta}(\sigma^{\rho\sigma})_{\alpha\beta} f_{\mu\nu} f_{\rho\sigma}\ , \tag{4.48}$$

where we used $(\sigma^{\mu\nu})_\alpha^{\ \beta} = \mathrm{tr}\ \sigma^{\mu\nu} = 0$. Furthermore,

$$(\sigma^{\mu\nu})^{\alpha\beta}(\sigma^{\rho\sigma})_{\alpha\beta} = \frac{1}{2}\left(g^{\mu\rho}g^{\nu\sigma} - g^{\mu\sigma}g^{\nu\rho}\right) - \frac{i}{2}\epsilon^{\mu\nu\rho\sigma} \tag{4.49}$$

(with $\epsilon^{0123} = +1$) so that

$$\int d^2\theta\ W^\alpha W_\alpha = -\frac{1}{2} f_{\mu\nu} f^{\mu\nu} - 2i\lambda\sigma^\mu\partial_\mu\overline{\lambda} + D^2 + \frac{i}{4}\epsilon^{\mu\nu\rho\sigma} f_{\mu\nu} f_{\rho\sigma}\ . \tag{4.50}$$

Note that the first three terms are real while the last one is purely imaginary.

Chapter 5

Supersymmetric gauge theories

We first discuss pure $N=1$ gauge theory which only involves the vector multiplet and will be described in terms of the vector superfield of the previous section. We will need a slight generalization of the definition of W_α to the non-abelian case. All members of the vector multiplet (the gauge boson v_μ and the gaugino λ) necessarily are in the same representation of the gauge group, i.e. in the adjoint representation. Lateron we will couple chiral multiplets to this vector multiplet. The chiral fields can be in any representation of the gauge group, e.g. in the fundamental one.

5.1 Pure $N=1$ gauge theory

We start with the vector multiplet (4.36) with every component now in the adjoint representation of the gauge group G, i.e. $V \equiv V_a T^a,\ a = 1, \dots \mathrm{dim}G$ where the T_a are the generators in the adjoint. The basic object then is e^V rather than V itself. The non-abelian generalisation of the transformation (4.37) is now

$$e^V \to e^{i\Lambda^\dagger} e^V e^{-i\Lambda} \ \Leftrightarrow e^{-V} \to e^{i\Lambda} e^{-V} e^{-i\Lambda^\dagger} \tag{5.1}$$

with Λ a chiral superfield. To first order in Λ this reproduces (4.37) with $\phi = -i\Lambda$. We will construct an action such that this non-linear transformation is a (local) symmetry. This transformation can again be used to set χ, C, M, N and one component of v_μ to zero, resulting in the same component expansion (4.39) of V in the Wess-Zumino gauge. From now on we adopt this WZ gauge. Then $V^n = 0, n \geq 3$. The same remains true if some D_α or $\overline{D}_{\dot\alpha}$ are inserted in the product, e.g. $V(D_\alpha V)V = 0$. One simply has

$$e^V = 1 + V + \frac{1}{2} V^2 \ . \tag{5.2}$$

The superfields W_α are now defined as

$$W_\alpha = -\frac{1}{4}\overline{DD}\left(e^{-V}D_\alpha e^V\right) \quad , \quad \overline{W}_{\dot\alpha} = +\frac{1}{4}DD\left(e^V\overline{D}_{\dot\alpha}e^{-V}\right) \ , \tag{5.3}$$

which to first order in V reduces to the abelian definition (4.41). Under the transformation (5.1) one then has

$$\begin{aligned} W_\alpha &\to -\tfrac{1}{4}\overline{DD}\left(e^{i\Lambda}e^{-V}e^{-i\Lambda^\dagger}D_\alpha\left(e^{i\Lambda^\dagger}e^V e^{-i\Lambda}\right)\right) \\ &= -\tfrac{1}{4}\overline{DD}\left(e^{i\Lambda}e^{-V}\left((D_\alpha e^V)e^{-i\Lambda} + e^V D_\alpha e^{-i\Lambda}\right)\right) \ . \end{aligned} \tag{5.4}$$

The second term is $-\frac{1}{4}\overline{DD}\left(e^{i\Lambda}D_\alpha e^{-i\Lambda}\right)$ and vanishes for the same reason as $\frac{1}{4}\overline{DD}D_\alpha\phi$ in (4.42). Thus

$$W_\alpha \to -\frac{1}{4}e^{i\Lambda}\overline{DD}\left(e^{-V}D_\alpha e^V\right)e^{-i\Lambda} = e^{i\Lambda}W_\alpha e^{-i\Lambda} \tag{5.5}$$

i.e. W_α transforms covariantly under (5.1). Similarly, one has

$$\overline{W}_{\dot\alpha} \to e^{i\Lambda^\dagger}W_\alpha e^{-i\Lambda^\dagger} \ . \tag{5.6}$$

Next, we want to obtain the component expansion of W_α in WZ gauge. Inserting the expansion (5.2) into the definition (5.3) gives

$$W_\alpha = -\frac{1}{4}\overline{DD}D_\alpha V + \frac{1}{8}\overline{DD}[V, D_\alpha V] \ . \tag{5.7}$$

The first term is the same as in the abelian case and has been computed in (4.46), while for the new term we have (all arguments are y^μ)

$$[V, D_\alpha V] = \overline{\theta\theta}(\sigma^{\nu\mu}\theta)_\alpha[v_\mu, v_\nu] + i\theta\theta\overline{\theta\theta}\sigma^\mu_{\alpha\dot\beta}[v_\mu, \overline{\lambda}^{\dot\beta}] \tag{5.8}$$

and then, using again $\overline{DD\theta\theta} = -4$

$$\frac{1}{8}\overline{DD}[V, D_\alpha V] = \frac{1}{2}(\sigma^{\mu\nu}\theta)_\alpha[v_\mu, v_\nu] - \frac{i}{2}\theta\theta\sigma^\mu_{\alpha\dot\beta}[v_\mu, \overline{\lambda}^{\dot\beta}] \ . \tag{5.9}$$

Adding this to (4.46) simply turns ordinary derivatives of the fields into gauge covariant derivatives and we finally obtain

$$W_\alpha = -i\lambda_\alpha(y) + \theta_\alpha D(y) + i(\sigma^{\mu\nu}\theta)_\alpha F_{\mu\nu}(y) + \theta\theta(\sigma^\mu D_\mu\overline{\lambda}(y))_\alpha \tag{5.10}$$

where now

$$F_{\mu\nu} = \partial_\mu v_\nu - \partial_\nu v_\mu - \frac{i}{2}[v_\mu, v_\nu] \tag{5.11}$$

and

$$D_\mu\overline{\lambda} = \partial_\mu\overline{\lambda} - \frac{i}{2}[v_\mu, \overline{\lambda}] \ . \tag{5.12}$$

The reader should not confuse the gauge covariant derivative D_μ neither with the super covariant derivatives D_α and $\overline{D}_{\dot\alpha}$, nor with the auxiliary field D.

The gauge group generators T^a satisfy

$$[T^a, T^b] = i f^{abc} T^c \tag{5.13}$$

with real structure constants f^{abc}. The field strength then is $F^a_{\mu\nu} = \partial_\mu v^a_\nu - \partial_n v^a_\mu + \frac{1}{2} f^{abc} v^b_\mu v^c_\nu$. We introduce the gauge coupling constant g by scaling the superfield V and hence all of its component fields as

$$V \to 2g\, V \quad \Leftrightarrow v_\mu \to 2g\, v_\mu \ , \ \lambda \to 2g\, \lambda \ , \ D \to 2g\, D \tag{5.14}$$

so that then we have the rescaled definitions of gauge covariant derivative and field strength

$$\begin{aligned} D_\mu \lambda = \partial_\mu \lambda - ig[v_\mu, \lambda] \quad &\Rightarrow \quad (D_\mu\lambda)^a = \partial_\mu \lambda^a + g f^{abc} v^b_\mu \lambda^c \\ F_{\mu\nu} = \partial_\mu v_\nu - \partial_\nu v_\mu - ig[v_\mu, v_\nu] \quad &\Rightarrow \quad F^a_{\mu\nu} = \partial_\mu v^a_\nu - \partial_\nu v^a_\mu + g f^{abc} v^b_\mu v^c_\nu \ . \end{aligned} \tag{5.15}$$

(We have implicitly assumed that the gauge group is simple so that there is a single coupling constant g. The generalisation to $G = G_1 \times G_2 \times \ldots$ and several $g_1, \ g_2, \ldots$ is straightforward.) Then the component expansion (5.10) of W_α remains unchanged, except for two things: there is on overall factor $2g$ multiplying the r.h.s. and $F_{\mu\nu}$ and $D_\mu\lambda$ are now given by (5.15). It follows that (4.50) also remains unchanged except for the replacements $f_{\mu\nu} \to F_{\mu\nu}$ and $\partial_\mu\overline{\lambda} \to D_\mu\overline{\lambda}$ and an overall factor $4g^2$. One then introduces the complex coupling constant

$$\tau = \frac{\Theta}{2\pi} + \frac{4\pi i}{g^2} \tag{5.16}$$

where Θ stands for the Θ-angle. (We use a capital Θ to avoid confusion with the superspace coordinates θ.) Then

$$\begin{aligned} \mathcal{L}_{\text{gauge}} &= \tfrac{1}{32\pi} \text{Im} \left(\tau \int \text{d}^2\theta \ \text{Tr}\, W^\alpha W_\alpha\right) \\ &= \text{Tr}\left(-\tfrac{1}{4} F_{\mu\nu}F^{\mu\nu} - i\lambda\sigma^\mu D_\mu \overline{\lambda} + \tfrac{1}{2} D^2\right) + \tfrac{\Theta}{32\pi^2} g^2 \, \text{Tr} F_{\mu\nu}\widetilde{F}^{\mu\nu} \end{aligned} \tag{5.17}$$

where

$$\widetilde{F}^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma} F_{\rho\sigma} \tag{5.18}$$

is the dual field strength. The single term $\text{Tr}W^\alpha W_\alpha$ has produced both, the conventionally normalised gauge kinetic term $-\frac{1}{4}\text{Tr}F_{\mu\nu}F^{\mu\nu}$ and the instanton density $\frac{g^2}{32\pi^2}\text{Tr}F_{\mu\nu}\widetilde{F}^{\mu\nu}$ which multiplies the Θ-angle!

5.2 $N = 1$ gauge theory with matter

We now add chiral (matter) multiplets ϕ^i transforming in some representation R of the gauge group where the generators are represented by matrices $(T_R^a)^i{}_j$. Then

$$\phi^i \to \left(e^{i\Lambda}\right)^i{}_j \phi^j \quad , \quad \phi_i^\dagger \to \phi_j^\dagger \left(e^{-i\Lambda^\dagger}\right)^j{}_i \tag{5.19}$$

or simply $\phi \to e^{i\Lambda}\phi$, $\phi^\dagger \to \phi^\dagger e^{-i\Lambda^\dagger}$ where $\Lambda = \Lambda^a T_R^a$ is understood. Then

$$\phi^\dagger e^V \phi \equiv \phi^\dagger e^{V^a T_R^a} \phi \equiv \phi_i^\dagger \left(e^V\right)^i{}_j \phi^j \tag{5.20}$$

is the gauge invariant generalisation of the kinetic term and

$$\mathcal{L}_{\text{matter}} = \int \mathrm{d}^2\theta \mathrm{d}^2\overline{\theta}\ \phi^\dagger e^V \phi + \int \mathrm{d}^2\theta\ W(\phi) + \int \mathrm{d}^2\overline{\theta}\ [W(\phi)]^\dagger\ . \tag{5.21}$$

Note that we have not yet scaled V by $2g$, or equivalently we set $2g = 1$ for the time being to simplify the formula. We want to compute the $\theta\theta\overline{\theta\theta}$ component (D-term) of $\phi^\dagger e^V \phi = \phi^\dagger\phi + \phi^\dagger V \phi + \frac{1}{2}\phi^\dagger V^2 \phi$. The first term is given by (4.29). The second term is

$$\begin{aligned} \phi^\dagger V \phi\Big|_{\theta\theta\overline{\theta\theta}} &= \tfrac{i}{2} z^\dagger v^\mu \partial_\mu z - \tfrac{i}{2}\partial_\mu z^\dagger v^\mu z - \tfrac{1}{2}\overline{\psi}\overline{\sigma}^\mu v_\mu \psi \\ &+ \tfrac{i}{\sqrt{2}} z^\dagger \lambda\psi - \tfrac{i}{\sqrt{2}}\overline{\psi\lambda} z + \tfrac{1}{2} z^\dagger D z \end{aligned} \tag{5.22}$$

and the third term is

$$\phi^\dagger V^2 \phi\Big|_{\theta\theta\overline{\theta\theta}} = \frac{1}{4} z^\dagger v^\mu v_\mu z\ . \tag{5.23}$$

Combining all three terms gives

$$\begin{aligned} \phi^\dagger e^V \phi\Big|_{\theta\theta\overline{\theta\theta}} &= (D_\mu z)^\dagger D^\mu z - i\overline{\psi}\overline{\sigma}^\mu D_\mu \psi + f^\dagger f \\ &+ \tfrac{i}{\sqrt{2}} z^\dagger \lambda\psi - \tfrac{i}{\sqrt{2}}\overline{\psi\lambda} z + \tfrac{1}{2} z^\dagger D z + \text{total derivative}\ . \end{aligned} \tag{5.24}$$

with $D_\mu z = \partial_\mu z - \frac{i}{2} v_\mu^a T_R^a z$ and $D_\mu \psi = \partial_\mu \psi - \frac{i}{2} v_\mu^a T_R^a \psi$. We now rescale $V \to 2gV$ and use the first identity (2.15) to rewrite $\overline{\psi}\overline{\sigma}^\mu D_\mu \psi = \psi\sigma^\mu D_\mu \overline{\psi} + \text{total derivative}$. Then this is replaced by

$$\begin{aligned} \phi^\dagger e^{2gV} \phi\Big|_{\theta\theta\overline{\theta\theta}} &= (D_\mu z)^\dagger D^\mu z - i\psi\sigma^\mu D_\mu \overline{\psi} + f^\dagger f \\ &+ i\sqrt{2} g z^\dagger \lambda\psi - i\sqrt{2} g \overline{\psi\lambda} z + g z^\dagger D z + \text{total derivative}\ . \end{aligned} \tag{5.25}$$

now with $D_\mu z = \partial_\mu z - igv_\mu^a T_R^a z$ and $D_\mu \psi = \partial_\mu \psi - igv_\mu^a T_R^a \psi$. This part of the Lagrangian contains the kinetic terms for the scalar fields z^i and the matter fermions ψ^i, as well as specific interactions between the z^i, the ψ^i and the gauginos λ^a. One has e.g. $z^\dagger \lambda \psi \equiv z_i^\dagger (T_R^a)^i{}_j \lambda^a \psi^j$.

What happens to the superpotential $W(\phi)$? This must be a chiral superfield and hence must be constructed from the ϕ^i alone. It must also be gauge invariant which imposes severe constraints on the superpotential. A term of the form $a_{i_1,\ldots i_n} \phi^{i_1} \ldots \phi^{i_n}$ will only be allowed if the n-fold product of the representation R contains the trivial representation and then $a_{i_1,\ldots i_n}$ must be an invariant tensor of the gauge group. An example is $G = \mathrm{SU}(3)$ with $R = \mathbf{3}$. Then $\mathbf{3} \times \mathbf{3} \times \mathbf{3} = \mathbf{1} + \ldots$ and the correponding SU(3) invariant tensor is ϵ_{ijk}. In this example, however, bilinears would not be gauge invariant. On the other hand, the representation R need not be irreducible. Taking again the example of $G = \mathrm{SU}(3)$ one may have $R = \mathbf{3} \oplus \overline{\mathbf{3}}$ corresponding to a chiral superfield ϕ^i transforming as $\mathbf{3}$ ("quark") and a chiral superfield $\widetilde{\phi}_i$ transforming as $\overline{\mathbf{3}}$ ("antiquark"). Then one can form the gauge invariant chiral superfield $\widetilde{\phi}_i \phi^i$ which corresponds to a "quark" mass term.

There is a last type of term that may appear in case the gauge group simply is U(1) or contains U(1) factors.footnote If there is at least an extra U(1) factor the gauge group certainly is not simple and we have several coupling constants: These are the Fayet-Iliopoulos terms. Let V^A denote the vector superfield in the abelian case, or the component corresponding to an abelian factor. Then under an abelian gauge transformation, $V^A \to V^A - i\Lambda + i\Lambda^\dagger$, with Λ a chiral superfield. From the component expansion of such a chiral or anti chiral superfield (4.20) or (4.21) one sees that the D-term (the term $\sim \theta\theta\overline{\theta}\overline{\theta}$) transforms as $D^A \to D^A + \partial_\mu \partial^\mu (\ldots)$, i.e. as a total derivative. Being a D-term, it also transforms as a total derivative under susy. It follows that

$$\mathcal{L}_{\mathrm{FI}} = \sum_{A \in \text{abelian factors}} \xi^A \int \mathrm{d}^2\theta \mathrm{d}^2\overline{\theta}\ V^A = \frac{1}{2} \sum_{A \in \text{abelian factors}} \xi^A D^A \tag{5.26}$$

is a susy and gauge invariant Lagrangian (i.e. up to total derivatives).

We can finally write the full $N = 1$ Lagrangian, being the sum of (5.17), (5.25) and (5.26):

$$\begin{aligned}
\mathcal{L} &= \mathcal{L}_{\text{gauge}} + \mathcal{L}_{\text{matter}} + \mathcal{L}_{\text{FI}} \\
&= \tfrac{1}{32\pi}\text{Im}\,(\tau \textstyle\int \text{d}^2\theta\ \text{Tr}\, W^\alpha W_\alpha) + 2g\sum_A \xi^A \int \text{d}^2\theta \text{d}^2\overline{\theta}\ V^A \\
&+ \textstyle\int \text{d}^2\theta \text{d}^2\overline{\theta}\ \phi^\dagger e^{2gV}\phi + \int \text{d}^2\theta\ W(\phi) + \int \text{d}^2\overline{\theta}\ [W(\phi)]^\dagger \\
&= \text{Tr}\left(-\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} - i\lambda\sigma^\mu D_\mu\overline{\lambda} + \tfrac{1}{2}D^2\right) + \tfrac{\Theta}{32\pi^2}g^2\,\text{Tr}F_{\mu\nu}\widetilde{F}^{\mu\nu} + g\textstyle\sum_A \xi^A D^A \\
&+ (D_\mu z)^\dagger D^\mu z - i\psi\sigma^\mu D_\mu\overline{\psi} + f^\dagger f + i\sqrt{2}gz^\dagger\lambda\psi - i\sqrt{2}g\overline{\psi\lambda}z + gz^\dagger Dz \\
&- \tfrac{\partial W}{\partial z^i}f^i + h.c. - \tfrac{1}{2}\tfrac{\partial^2 W}{\partial z^i\partial z^j}\psi^i\psi^j + h.c. + \text{total derivative}\ .
\end{aligned} \tag{5.27}$$

The auxiliary field equations of motion are

$$f_i^\dagger = \frac{\partial W}{\partial z^i} \tag{5.28}$$

and

$$D^a = -gz^\dagger T^a z - g\xi^a \tag{5.29}$$

where it is understood that $\xi^a = 0$ if a does not take values in an abelian factor of the gauge group. Substituting this back into the Lagrangian one finds

$$\begin{aligned}
\mathcal{L} &= \text{Tr}\left(-\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} - i\lambda\sigma^\mu D_\mu\overline{\lambda}\right) + \tfrac{\Theta}{32\pi^2}g^2\,\text{Tr}F_{\mu\nu}\widetilde{F}^{\mu\nu} \\
&+ (D_\mu z)^\dagger D^\mu z - i\psi\sigma^\mu D_\mu\overline{\psi} \\
&+ i\sqrt{2}gz^\dagger\lambda\psi - i\sqrt{2}g\overline{\psi\lambda}z - \tfrac{1}{2}\tfrac{\partial^2 W}{\partial z^i\partial z^j}\psi^i\psi^j - \tfrac{1}{2}\left(\tfrac{\partial^2 W}{\partial z^i\partial z^j}\right)^\dagger \overline{\psi}^i\overline{\psi}^j \\
&- V(z^\dagger, z) + \text{total derivative}\ ,
\end{aligned} \tag{5.30}$$

where the scalar potential $V(z^\dagger, z)$ is given by

$$V(z^\dagger, z) = f^\dagger f + \frac{1}{2}D^2 = \sum_i \left|\frac{\partial W}{\partial z^i}\right|^2 + \frac{g^2}{2}\sum_a \left|z^\dagger T^a z + \xi^a\right|^2\ . \tag{5.31}$$

5.3 Supersymmetric QCD

At this point we have all the ingredients to write the action for supersymmetric QCD. The gauge group is SU(3). (More generally, one could consider SU(N).) There are then gauge bosons v_μ^a, $a = 1, \ldots 8$ called the gluons, as well as their supersymmetric partners, the 8 gauginos or gluinos λ^a. If one considers pure

$N = 1$ "glue", this is all there is. To describe $N = 1$ QCD however, one also has to add quarks transforming in the **3** of SU(3) as well as antiquarks in the $\overline{\mathbf{3}}$. They are associated with chiral superfields. More precisely there are chiral superfields $Q^i_L = q^i_L + \sqrt{2}\theta\psi^i_L - \theta\theta f^i_L$, $i = 1, 2, 3$, and $L = 1, \ldots N_f$ labels the flavours. These fields transform in the **3** representation of the gauge group and correspond to left-handed quarks (or right-handed antiquarks). There are also chiral superfields $\widetilde{Q}_{i,L} = \widetilde{q}_{iL} + \sqrt{2}\theta\widetilde{\psi}_{iL} - \theta\theta\widetilde{f}_{iL}$, $i = 1, 2, 3$ and $L = 1, \ldots N_f$. They transform in the $\overline{\mathbf{3}}$ representation of the gauge group and correspond to left-handed antiquarks (or right-handed quarks).

Note that the gauge group does not contain any U(1) factor, and hence no Fayet-Iliopoulos term can appear. The component Lagrangian for massless susy QCD then is given by (5.30) with $z = (q, \widetilde{q})$, $\xi^a = 0$ and vanishing superpotential. Since all terms in the Lagrangian are diagonal in the flavour indices of the quarks and separately in the flavour indices of the antiquarks, there is an $\mathrm{SU}(N_f)_L \times \mathrm{SU}(N_f)_R$ global symmetry. In addition there is a $\mathrm{U}(1)_V$ acting as $Q \to e^{iv}Q$, $\widetilde{Q} \to e^{-iv}\widetilde{Q}$, as well as an $\mathrm{U}(1)_A$ acting as $Q \to e^{ia}Q$, $\widetilde{Q} \to e^{ia}\widetilde{Q}$. We also have the global $\mathrm{U}(1)_{\mathcal{R}}$ symmetry acting as $Q(x, \theta) \to e^{-iq}Q(x, e^{iq}\theta)$, $\widetilde{Q}(x, \theta) \to e^{-iq}\widetilde{Q}(x, e^{iq}\theta)$ and $V(x, \theta, \overline{\theta}) \to V(x, e^{iq}\theta, e^{-iq}\overline{\theta})$. Thus the global symmetry group in the massless case is $\mathrm{U}(N_f)_L \times \mathrm{U}(N_f)_R \times \mathrm{U}(1)_{\mathcal{R}}$

Due to the presence of both representations **3** and $\overline{\mathbf{3}}$ of the gauge group, one may add a gauge invariant superpotential

$$W(Q, \widetilde{Q}) = m_{L,M} Q^i_L \widetilde{Q}_{i,M} \ . \tag{5.32}$$

This is a quark mass term and $m_{L,M}$ is the $N_f \times N_f$ mass matrix. Using the global symmetry of the other terms in the Lagrangian (which is just the massless Lagrangian) one can diagonalise the superpotential so that it reads

$$W(Q, \widetilde{Q}) = \sum_L m_L Q^i_L \widetilde{Q}_{i,L} \ . \tag{5.33}$$

For the gauge group SU(3) one could also add the gauge invariant terms $a_{LMN}\epsilon_{ijk}Q^i_L Q^j_M Q^k_N$ and $\widetilde{a}_{LMN}\epsilon^{ijk}\widetilde{Q}_{iL}\widetilde{Q}_{jM}\widetilde{Q}_{kN}$. However, they explicitly violate baryon number conservation and will not be considered. Then finally one arrives at the following Lagrangian (where we suppress as much as possible all gauge and flavour indices):

$$\begin{aligned}
\mathcal{L} \;=\;& \mathrm{Tr}\left(-\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} - i\lambda\sigma^\mu D_\mu\overline{\lambda}\right) + \tfrac{\Theta}{32\pi^2}g^2\,\mathrm{Tr}F_{\mu\nu}\widetilde{F}^{\mu\nu} \\
+\;& (D_\mu q)^\dagger D^\mu q + (D_\mu\widetilde{q})^\dagger D^\mu\widetilde{q} - i\psi\sigma^\mu D_\mu\overline{\psi} - i\widetilde{\psi}\sigma^\mu D_\mu\overline{\widetilde{\psi}} \\
+\;& i\sqrt{2}gq^\dagger\lambda\psi + i\sqrt{2}g\widetilde{q}^\dagger\lambda\widetilde{\psi} - i\sqrt{2}g\overline{\psi\lambda}q - i\sqrt{2}g\overline{\widetilde{\psi}}\overline{\lambda}\widetilde{q} \\
-\;& \tfrac{1}{2}\textstyle\sum_L m_L\left(\psi_L\psi_L + \overline{\widetilde{\psi}}_L\overline{\widetilde{\psi}}_L\right) - V(q, \widetilde{q}, q^\dagger, \widetilde{q}^\dagger) + \text{total derivative}\ ,
\end{aligned} \tag{5.34}$$

where the scalar potential V is given by

$$V(q,\widetilde{q},q^\dagger,\widetilde{q}^\dagger) = \sum_{L=1}^{N_f} m_L^2 \left(q_L^\dagger q_L + \widetilde{q}_L^\dagger \widetilde{q}_L \right) + \frac{g^2}{2} \sum_{a=1}^{8} \left| q^\dagger T^a q + \widetilde{q}^\dagger T^a \widetilde{q} \right|^2 \ . \qquad (5.35)$$

Chapter 6

Spontaneously broken supersymmetry

6.1 Vacua in susy theories

Perturbation theory should be performed around a stable configuration. If quantum field theory is formulated using a euclidean functional integral, stable configurations correspond to minima of the euclidean action. A vacuum is a Lorentz invariant stable configuration. Lorentz invariance implies that all space-time derivatives and all fields that are nor scalars must vanish. Hence only scalar fields z^i can have a non-vanishing value in a vacuum configuration, i.e. a non-vanishing vacuum expectation value (vev), denoted by $\langle z^i \rangle$. Minimality of the euclidean action (or else minimality of the energy functional) then is equivalent to the scalar potential V having a minimum. Thus we have for a vacuum

$$\langle v_\mu^a \rangle = \langle \lambda^a \rangle = \langle \psi^i \rangle = \partial_\mu \langle z^i \rangle = 0 \quad , \quad V(\langle z^i \rangle, \langle z_i^\dagger \rangle) = \text{minimum} \; . \tag{6.1}$$

The minimum may be the global minimum of V in which case one has the true vacuum, or it may be a local minimum in which case one has a false vacuum that will eventually decay by quantum tunneling into the true vacuum (although the life-time may be extremely long). For a false or true vacuum one certainly has

$$\frac{\partial V}{\partial z^i}(\langle z^j \rangle, \langle z_j^\dagger \rangle) = \frac{\partial V}{\partial z_i^\dagger}(\langle z^j \rangle, \langle z_j^\dagger \rangle) = 0 \; . \tag{6.2}$$

This shows again that a vacuum is indeed a solution of the equations of motion.

Now in a supersymmetric theory the scalar potential is given by (5.31), namely

$$V(z, z^\dagger) = f_i^\dagger f^i + \frac{1}{2} D^a D^a \tag{6.3}$$

where

$$f_i^\dagger = \frac{\partial W(z)}{\partial z^i} \tag{6.4}$$

and

$$D^a = -g^a \left(z_i^\dagger (T^a)^i{}_j z^j + \xi^a \right) \tag{6.5}$$

where we allowd for Fayet-Iliopoulos terms $\sim \xi^a$ associated with possible U(1) factors and couplings g^a. Of course within each simple factor of the gauge group G the g^a are the same.[1] The potential (6.3) is non-negative and it will certainly be at its global minimum, namely $V = 0$, if

$$f^i(\langle z^\dagger \rangle) = D^a(\langle z \rangle, \langle z^\dagger \rangle) = 0 \ . \tag{6.6}$$

However, this system of equations does not necessarily have a solution as a simple counting argument shows: there are as many equations $f^i = 0$ as unknown $\langle z_i^\dagger \rangle$ (and as many complex conjugate equations $f_i^\dagger = 0$ as complex conjugate $\langle z^i \rangle$). On top of these there are dim$G$ equations $D^a = 0$ to be satisfied. We now have two cases.

a) If the equations (6.6) have a solution, then this solution is a global minimum of V (since $V = 0$) and hence a stable true vacuum. There can be many such solutions and then we have many degenerate vacuua. In addition to this true vacuum there can be false vacua satisfying (6.2) but not (6.6).

b) If the equations (6.6) have no solutions, the scalar potential V can never vanish and its minimum is strictly positive: $V \geq V_0 > 0$. Now a vacuum with strictly positive energy necessarily breaks supersymmetry. This means that the vacuum cannot be invariant under all susy generators. The proof is very simple: as in (3.4) we have for any state $|\Omega\rangle$

$$\langle \Omega | P_0 | \Omega \rangle = \frac{1}{4} || Q_\alpha | \Omega \rangle ||^2 + \frac{1}{4} || Q_\alpha^\dagger | \Omega \rangle ||^2 = 0 \ . \tag{6.7}$$

Now assume that $|\Omega\rangle$ is invariant under all susy generators, i.e. $Q_\alpha |\Omega\rangle = 0$. Then necessarily $\langle \Omega | P_0 | \Omega \rangle = 0$, and conversely if $\langle \Omega | P_0 | \Omega \rangle > 0$ not all Q_α and $Q_\alpha^\dagger$ can annihilate the state $|\Omega\rangle$. It is not surprising that an excited state, e.g. a one-particle state is not invariant under susy: indeed this is how susy transforms the different particles of a supermultiplet into each other. Non-invariance of the vacuum state has a different meaning: it implies that susy is really broken in the perturbation theory based on this vacuum. As usual, this is called spontaneous breaking of the (super)symmetry.

There is also another way to see that susy is broken if either $f^i(\langle z^\dagger \rangle) \neq 0$ or $D^a(\langle z \rangle, \langle z^\dagger \rangle) \neq 0$. Looking at the susy transformations of the fields one has from

[1] If $G = G_1 \times \ldots \times G_k \times \mathrm{U}(1) \times \ldots \times \mathrm{U}(1)$ with simple factors G_l of dimension d_l it is understood that $g^1 = \ldots = g^{d_1}$, $g^{d_1+1} = \ldots = g^{d_1+d_2}$, etc and $\xi^1 = \ldots \xi^{d_1+\ldots d_k} = 0$.

(4.24)

$$\begin{aligned} \delta\langle z^i\rangle &= \sqrt{2}\epsilon\langle\psi^i\rangle \\ \delta\langle\psi^i\rangle &= \sqrt{2}i\partial_\mu\langle z^i\rangle\sigma^\mu\overline{\epsilon} - \sqrt{2}\langle f^i\rangle\epsilon \\ \delta\langle f^i\rangle &= \sqrt{2}i\partial_\mu\langle\psi^i\rangle\sigma^\mu\overline{\epsilon} \end{aligned} \tag{6.8}$$

which upon taking into account (6.2) reduces to

$$\begin{aligned} \delta\langle z^i\rangle &= 0 \\ 0 = \delta\langle\psi^i\rangle &= -\sqrt{2}\langle f^i\rangle\epsilon \\ \delta\langle f^i\rangle &= 0 \end{aligned} \tag{6.9}$$

which can be consistent only if $\langle f^i\rangle \equiv f^i(\langle z^\dagger\rangle) = 0$. The argument similarly shows that $\delta\langle\lambda^a\rangle = 0$ is only possible if $\langle D^a\rangle \equiv D^a(\langle z\rangle, \langle z^\dagger\rangle) = 0$. More generally, a necessary condition for unbroken susy is that the susy variations of the fermions vanish in the vacuum.

6.2 The Goldstone theorem for susy

Goldstone's theorem states that whenever a continuous global symmetry is spontaneously broken, i.e. the vacuum is not invariant, there is a massless mode in the spectrum, i.e. a massless particle. The quantum numbers carried by the Goldstone particle are related to the broken symmetry. Similarly, we will show that if supersymmetry is spontaneously broken there is a massless spin one-half particle, i.e. a massless spinorial mode, sometimes called the Goldstino.

As we have seen, a vacuum that breaks susy is such that $\frac{\partial V}{\partial z^i}(\langle z^i\rangle, \langle z_i^\dagger\rangle) = 0$ (it is a vacuum) and[2] $\langle f^i\rangle \neq 0$ or $\langle D^a\rangle \neq 0$. Now from (6.3)-(6.5) we have

$$\frac{\partial V}{\partial z^i} = f^j\frac{\partial^2 W}{\partial z^i\partial z^j} - g^a D^a z_j^\dagger (T^a)^j{}_i \tag{6.10}$$

and this must vanish for any vacuum. We will combine this with the statement of gauge invariance of the superpotential W which reads

$$0 = \delta^{(a)}_{\text{gauge}} W = \frac{\partial W}{\partial z^i}\delta^{(a)}_{\text{gauge}} z^i = f_i^\dagger (T^a)^i{}_j z^j \ . \tag{6.11}$$

We can now combine the vanishing of (6.10) in the vacuum with the vev of the complex conjugate equation (6.11) into the matrix equation

$$M = \begin{pmatrix} \langle\frac{\partial^2 W}{\partial z^i\partial z^j}\rangle & -g^a\langle z_l^\dagger\rangle(T^a)^l{}_i \\ -g^b\langle z_l^\dagger\rangle(T^b)^l{}_j & 0 \end{pmatrix} \quad , \quad M\begin{pmatrix}\langle f^j\rangle \\ \langle D^a\rangle\end{pmatrix} = 0 \tag{6.12}$$

[2] As before, $\langle f^i\rangle$ is shorthand for $f^i(\langle z^\dagger\rangle)$ and $\langle D^a\rangle$ shorthand for $D^a(\langle z\rangle, \langle z^\dagger\rangle)$.

stating that the matrix appearing here has a zero eigenvalue. But this matrix exactly is the fermion mass matrix. Indeed, the non-derivative fermion bilinears in the Lagrangian (5.27) give rise in the vacuum to the following mass terms

$$\begin{aligned} & \left(i\sqrt{2}g^a\langle z_j^\dagger\rangle(T^a)^j{}_i\lambda^a\psi^i - \tfrac{1}{2}\langle\tfrac{\partial^2 W}{\partial z^i\partial z^j}\rangle\psi^i\psi^j\right) + h.c. \\ = & -\tfrac{1}{2}\left(\psi^i, \sqrt{2}i\lambda^b\right) M \begin{pmatrix} \psi^j \\ \sqrt{2}i\lambda^a \end{pmatrix} + h.c. \end{aligned} \tag{6.13}$$

with the same matrix M as defined in (6.12). This matrix has a zero eigenvalue, and this means that there is a zero mass fermion: the Goldstone fermion or Goldstino.

6.3 Mechanisms for susy breaking

We have seen that a minimum of V with $\langle f^i\rangle \neq 0$ or $\langle D^a\rangle \neq 0$ is a vacuum that breaks susy. This can be a true or false vacuum. If there is *no* vacuum with $\langle f^i\rangle = \langle D^a\rangle = 0$, i.e. no solution $\langle z^i\rangle$ to these equations, supersymmetry is necessarily broken by any vacuum. Whether or not there are solutions depends on the choice of superpotential W and whether the Fayet-Iliopoulos parameters ξ^a vanish or not.

6.3.1 O'Raifeartaigh mechanism

Assume first that no U(1) factors are present or else that the ξ^a vanish. Susy will be broken if $\frac{\partial W}{\partial z^i} = 0$ and $z_j^\dagger (T^a)^j{}_l\, z^l = 0$ have no solution. If the superpotential W has no linear term, $\langle z^i\rangle = 0$ will always be a solution. So let's assume that there is a linear term $W = a_i z^i + \ldots$. But this can be gauge invariant only if the representation R carried by the z^i contains at least one singlet, say $z^1 = Y$. As a simple example take

$$W = Y(a - X^2) + bZX + w(X, z^i) \tag{6.14}$$

with X, Y, Z all singlets. Then $f_Y^\dagger = \frac{\partial W}{\partial Y} = a - X^2$ and $f_Z^\dagger = \frac{\partial W}{\partial Z} = bX$ cannot both vanish so that there is no susy preserving vacuum solution.

6.3.2 Fayet-Iliopoulos mechanism

Let there now be at least one U(1) and non-vanishing ξ. The relevant part of $D = 0$ is

$$0 = \sum_i q_i |z^i|^2 + \xi \tag{6.15}$$

where the q_i are the U(1) charges of z^i. If all charges q_i had the same sign, taking a ξ of the same sign as the q_i would forbid the existence of solutions and break susy. However, absence of chiral anomalies for the U(1) imposes $\sum_i q_i^3 = 0$ so that charges of both signs must be present and there is always a solution to (6.15). One needs further constraints from $f^i = 0$ to break susy. To see how this works consider again a simple model. Take two chiral multiplets ϕ^1 and ϕ^2 with charges $q_1 = -q_2 = 1$ so that (6.15) reads $|z^1|^2 - |z^2|^2 + \xi = 0$ and take a superpotential $W = m\phi^1\phi^2$. Then $f^1 = mz_2^\dagger$ and $f^2 = mz_1^\dagger$ and clearly, if $m \neq 0$ and $\xi \neq 0$, we cannot simultaneously have $f^1 = f^2 = D = 0$ so that susy will be broken.

6.4 Mass formula

If supersymmetry is unbroken all particles within a supermultiplet have the same mass. Although this will no longer be true if supersymmetry is (spontaneously) broken, but one can still relate the differences of the squared masses to the susy breaking parameters $\langle f^i \rangle$ and $\langle D^a \rangle$.

Let us derive the masses of the different particles: vectors, fermions and scalars. We begin with the vector fields. In the presence of non-vanishing vevs of the scalars, some or all of the vector gauge fields will become massive by the Higgs mechanism. Indeed the term $(D_\mu z^i)^\dagger (D^\mu z^i)$ present in the Lagrangian (5.27) gives rise to a mass term $g^2 \langle z^\dagger T^a T^b z \rangle v_\mu^a v^{a\mu}$, while the gauge kinetic term is normalised in the standard way. Thus the mass matrix for the spin-one fields is

$$\left(\mathcal{M}_1^2\right)^{ab} = 2g^2 \langle z^\dagger T^a T^b z \rangle \ . \tag{6.16}$$

It will be useful to introduce the notations

$$D_i^a = \frac{\partial D^a}{\partial z^i} = -g(z^\dagger T^a)_i \quad , \quad D^{ia} = \frac{\partial D^a}{\partial z_i^\dagger} = -g(T^a z)^i \tag{6.17}$$

as well as $D_j^{ai} = -gT_j^{ai}$, and similarly

$$f^{ij} = \frac{\partial f^i}{\partial z_j^\dagger} = \frac{\partial^2 \overline{W}}{\partial z_j^\dagger \partial z_i^\dagger} \quad , \quad f_{ij} = \frac{\partial f_i^\dagger}{\partial z^j} = \frac{\partial^2 W}{\partial z^j \partial z^i} \tag{6.18}$$

etc. Then eq. (6.16) can be written as

$$\left(\mathcal{M}_1^2\right)^{ab} = 2\langle D_i^a D^{bi}\rangle = 2\langle D_i^a\rangle\langle D^{bi}\rangle \ . \tag{6.19}$$

Next, for the spin-one-half fermions the mass matrix can be read from (6.12) and (6.13) or again directly from (5.27). The mass terms are

$$-\frac{1}{2}(\psi^i \quad \lambda^a)\mathcal{M}_{\frac{1}{2}}\begin{pmatrix}\psi^j\\ \lambda^b\end{pmatrix} + \text{h.c.} \quad , \quad \mathcal{M}_{\frac{1}{2}} = \begin{pmatrix}\langle f_{ij}\rangle & \sqrt{2}i\langle D_i^b\rangle\\ \sqrt{2}i\langle D_j^a\rangle & 0\end{pmatrix} \tag{6.20}$$

with the squared masses of the fermions being given by the eigenvalues of the hermitian matrix

$$\left(\mathcal{M}_{\frac{1}{2}}\mathcal{M}_{\frac{1}{2}}^\dagger\right) = \begin{pmatrix}\langle f_{il}\rangle\langle f^{jl}\rangle + 2\langle D_i^c\rangle\langle D^{cj}\rangle & -\sqrt{2}i\langle f_{il}\rangle\langle D^{bl}\rangle\\ \sqrt{2}i\langle D_l^a\rangle\langle f^{jl}\rangle & 2\langle D_l^a\rangle\langle D^{bl}\rangle\end{pmatrix} \ . \tag{6.21}$$

Finally for the scalars the mass terms are[3]

$$-\frac{1}{2}(z^i \quad z_j^\dagger)\mathcal{M}_0^2\begin{pmatrix}z_k^\dagger\\ z^l\end{pmatrix} \tag{6.22}$$

with

$$\mathcal{M}_0^2 = \begin{pmatrix}\langle\frac{\partial^2 V}{\partial z^i\partial z_k^\dagger}\rangle & \langle\frac{\partial^2 V}{\partial z^i\partial z^l}\rangle\\ \langle\frac{\partial^2 V}{\partial z_j^\dagger\partial z_k^\dagger}\rangle & \langle\frac{\partial^2 V}{\partial z_j^\dagger\partial z^l}\rangle\end{pmatrix} \ . \tag{6.23}$$

Using (6.3) one finds that this matrix equals

$$\begin{pmatrix}\langle f_{ip}\rangle\langle f^{kp}\rangle + \langle D^{ak}\rangle\langle D_i^a\rangle + \langle D^a\rangle D_i^{ak} & \langle f^p\rangle\langle f_{ilp}\rangle + \langle D_i^a\rangle\langle D_l^a\rangle\\ \langle f_p^\dagger\rangle\langle f^{jkp}\rangle + \langle D^{aj}\rangle\langle D^{ak}\rangle & \langle f_{lp}\rangle\langle f^{jp}\rangle + \langle D^{aj}\rangle\langle D_l^a\rangle + \langle D^a\rangle D_l^{aj}\end{pmatrix} \tag{6.24}$$

It is now straightforward to give the traces which yield the sums of the masses squared of the vectors, fermions and scalars, respectively.

$$\begin{aligned}\text{tr}\,\mathcal{M}_1^2 &= 2\langle D_i^a\rangle\langle D^{ai}\rangle\\ \text{tr}\,\mathcal{M}_{\frac{1}{2}}\mathcal{M}_{\frac{1}{2}}^\dagger &= \langle f_{il}\rangle\langle f^{il}\rangle + 4\langle D_i^a\rangle\langle D^{ai}\rangle\\ \text{tr}\,\mathcal{M}_0^2 &= 2\langle f_{ip}\rangle\langle f^{ip}\rangle + 2\langle D_i^a\rangle\langle D^{ai}\rangle - 2g\langle D^a\rangle\text{tr}\,T^a\end{aligned} \tag{6.25}$$

and

$$\text{Str}\mathcal{M}^2 \equiv 3\text{tr}\,\mathcal{M}_1^2 - 2\text{tr}\,\mathcal{M}_{\frac{1}{2}}\mathcal{M}_{\frac{1}{2}}^\dagger + \text{tr}\,\mathcal{M}_0^2 = -2g\langle D^a\rangle\text{tr}\,T^a \ . \tag{6.26}$$

[3] The way the z and $z^\dagger$ are grouped as well as the $\frac{1}{2}$ may seem peculiar at the first sight, but they are easily explained by the example of a single complex scalar field for which the mass term is $m^2 z z^\dagger$. Then simply $\mathcal{M}_0^2 = \begin{pmatrix}m^2 & 0\\ 0 & m^2\end{pmatrix}$ and (6.22) yields $-\frac{1}{2}zm^2z^\dagger - \frac{1}{2}z^\dagger m^2 z = -m^2 z^\dagger z$.

In this supertrace we have counted two degrees of freedom for spinors and three for vectors as appropriate in the massive case (the massless states do not contribute anyhow). We see that if $\langle D^a \rangle = 0$ or $\mathrm{tr}\, T^a = 0$ (no U(1) factor) this supertrace vanishes, stating that the sum of the squared masses of all bosonic degrees of freedom equals the sum for all fermionic ones. Without susy breaking this is a triviality. In the presence of susy breaking this supertrace formula is still a strong constraint on the mass spectrum. In particular, if susy is broken only by non-vanishing $\langle f^i \rangle$ (and $\langle f_i^\dagger \rangle$), or if all gauge group generators are traceless, one must still have $\mathrm{Str}\mathcal{M}^2 = 0$.

Consider e.g. susy QCD. The gauge group is SU(3) and $\mathrm{tr}\, T^a = 0$, while the gauge group must remain unbroken. Then $\mathcal{M}_1^2 = 0$ so that $\langle D_i^a \rangle = \langle D^{ai} \rangle = 0$. Note from (6.20) that it is then obvious that also the gauginos (gluinos) remain massless, while supertrace formula tells us that the sum of the masses squared of the scalar quarks must equal (twice) the sum for the quarks. This means that the scalar quarks cannot all be heavier than the heaviest quark, and some must be substantially lighter. Since no massless gluinos and relatively light scalar quarks have been found experimentally, this scenario seems to be ruled out by experiment. However, it would be too quick to conclude that one cannot embed QCD into a susy theory. Indeed, there are two ways out. First, the mass formula derived here only give the tree-level masses and are corrected by loop effects. Typically, one introduces one or several additional chiral multiplets which trigger the susy breaking. Through loop diagrams this susy breaking then propagates to the gauge theory we are interested in and, in principle, one can achieve heavy gauginos and heavy scalar quarks this way. leaving massless gauge fields and light fermions. Second, the susy theory may be part of a supergravity theory which is spontaneously broken, and in this case one rather naturally obtains experimentally reasonable mass relations.

Let's discuss a bit more the mass matrices derived above. As in the example of susy QCD just discussed, if the gauge symmetry is unbroken, $\mathcal{M}_1^2 = 0$ implying $\langle D_i^a \rangle = 0$, so that the fermion mass matrix reduces to $\left(\mathcal{M}_{\frac{1}{2}} \mathcal{M}_{\frac{1}{2}}^\dagger\right) = \begin{pmatrix} \langle f_{il} \rangle \langle f^{jl} \rangle & 0 \\ 0 & 0 \end{pmatrix}$, showing again that the gauginos are massless, too. If we now suppose that there are no Fayet-Iliopoulos parameters, $\langle D_i^a \rangle = 0$ implies that also $\langle D^a \rangle = 0$ as is easily seen[4] so that the scalar mass matrix now is

$$\mathcal{M}_0^2 = \begin{pmatrix} \langle f_{ip} \rangle \langle f^{kp} \rangle & \langle f^p \rangle \langle f_{ilp} \rangle \\ \langle f_p^\dagger \rangle \langle f^{jkp} \rangle & \langle f_{lp} \rangle \langle f^{jp} \rangle \end{pmatrix} . \tag{6.27}$$

The block diagonal terms are the same as for the fermions ψ^i, but the block

[4] One has $D_j^a D^{bj} = z^\dagger T^a T^b z$ and $D_j^{[a} D^{b]j} = \frac{i}{2} f^{abc} z^\dagger T^c z = -\frac{i}{2g} f^{abc} D^c$ (if there are no FI parameters). Thus if $\langle D_i^a \rangle = 0$, also $\langle D^{bj} \rangle = 0$ and this then implies $\langle D^c \rangle = 0$.

off-diagonal terms give an additional contribution

$$-\frac{1}{2}\langle f^p\rangle\langle f_{ilp}\rangle z^i z^l \ + \ \text{h.c.}\ . \tag{6.28}$$

The effect of this term typically is to lift the mass degeneracy between the real and the imaginary parts of the scalar fields, splitting the masses in a symmetric way with respect to the corresponding fermion masses. This is of course in agreement with $\mathrm{Str}\mathcal{M}^2 = 0$ in this case.

Chapter 7

The non-linear sigma model

As long as one wants to formulate a fundamental, i.e. microscopic theory, one is guided by the principle of renomalisability. For the theory of chiral superfields ϕ only this implies at most cubic superpotentials (leading to at most quartic scalar potentials) and kinetic terms $K^i{}_j \phi_i^\dagger \phi^j$ with some constant hermitian matrix K. After diagonalisation and rescaling of the fields this then reduces to the canonical kinetic term $\phi_i^\dagger \phi^i$. Thus we are back to the Wess-Zumino model studied above.

In many cases, however, the theory on considers is an *effective* theory, valid at low energies only. Then renormalisability no longer is a criterion. The only restriction for such a low-energy effective theory is to contain no more than two (space-time) derivatives. Higher derivative terms are irrelevant at low energies. Thus we are led to study the supersymmetric non-linear sigma model. Another motivation comes from supergravity which is not renormalisable anyway. We will first consider the model for chiral multiplets only, and then extend the resulting theory to a gauge invariant one.

7.1 Chiral multiplets only

We start with the action

$$S = \int \mathrm{d}^4x \left(\int \mathrm{d}^2\theta \mathrm{d}^2\overline{\theta}\ K(\phi^i, \phi_i^\dagger) + \int \mathrm{d}^2\theta\ w(\phi^i) + \int \mathrm{d}^2\overline{\theta}\ w^\dagger(\phi_i^\dagger) \right) \ . \tag{7.1}$$

We have denoted the superpotential by w rather than W. The function $K(\phi^i, \phi_i^\dagger)$ must be real superfield, which will be the case if $\overline{K}(z^i, z_j^\dagger) = K(z_i^\dagger, z^j)$. Derivatives with respect to its arguments will be denoted as

$$K_i = \frac{\partial}{\partial z^i} K(z, z^\dagger) \quad , \quad K^j = \frac{\partial}{\partial z_j^\dagger} K(z, z^\dagger) \quad , \quad K_i^j = \frac{\partial^2}{\partial z^i \partial z_j^\dagger} K(z, z^\dagger) \tag{7.2}$$

etc. (Note that one does not need to distinguish indices like ${K^i}_j$ or ${K_j}^i$ since the partial derivatives commute.) Similarly we have

$$w_i = \frac{\partial}{\partial z^i} w(z) \quad , \quad w_{ij} = \frac{\partial^2}{\partial z^i \partial z^j} w(z) \tag{7.3}$$

etc. We also use $w^i = [w_i]^\dagger$, $w^{ij} = [w_{ij}]^\dagger$.

The expansion of the F-terms in components was already given in (4.25). We may rewrite this as

$$w(\phi) = w(z) + w_i \Delta^i + \frac{1}{2} w_{ij} \Delta^i \Delta^j \tag{7.4}$$

with arguments y^μ understood and

$$\Delta^i(y) = \phi^i - z^i(y) = \sqrt{2}\theta\psi^i(y) - \theta\theta f^i(y) \ . \tag{7.5}$$

Then extracting the $\theta\theta$-components of Δ^i and $\Delta^i\Delta^j$ yields (4.25) again, i.e.

$$\int \mathrm{d}^2\theta \ w(\phi^i) + h.c. = \left(-w_i f^i - \frac{1}{2} w_{ij} \psi^i \psi^j \right) + h.c. \ . \tag{7.6}$$

The component expansion of the D-term is more involved, since now $\Delta^i(x) = \phi^i - z^i(x)$ and $\Delta^\dagger_i(x) = \phi^\dagger_i - z^\dagger_i(x)$ appear. We have from (4.20)

$$\begin{aligned} \Delta^j &= \sqrt{2}\theta\psi^j + i\theta\sigma^\mu\overline{\theta}\partial_\mu z^j - \theta\theta f^j - \tfrac{i}{\sqrt{2}}\theta\theta\partial_\mu\psi^j\sigma^\mu\overline{\theta} - \tfrac{1}{4}\theta\theta\overline{\theta\theta}\partial^2 z^j \\ \Delta^\dagger_j &= \sqrt{2}\overline{\theta\psi}_j - i\theta\sigma^\mu\overline{\theta}\partial_\mu z^\dagger_j - \overline{\theta\theta} f^\dagger_j + \tfrac{i}{\sqrt{2}}\overline{\theta\theta}\theta\sigma^\mu\partial_\mu\overline{\psi}_j - \tfrac{1}{4}\theta\theta\overline{\theta\theta}\partial^2 z^\dagger_j \end{aligned} \tag{7.7}$$

with all fields having x^μ as argument. Note that $\Delta^i\Delta^j\Delta^k = \Delta^\dagger_i\Delta^\dagger_j\Delta^\dagger_k = 0$ so that at most two Δ and two $\Delta^\dagger$ can appear in the expansion. One has the Taylor expansion of $K(\phi^i, \phi^\dagger_i)$

$$\begin{aligned} K(\phi^i, \phi^\dagger_i) &= K(z^i, z^\dagger_i) + K_i\Delta^i + K^i\Delta^\dagger_i + \tfrac{1}{2}K_{ij}\Delta^i\Delta^j + \tfrac{1}{2}K^{ij}\Delta^\dagger_i\Delta^\dagger_j + K^j_i\Delta^i\Delta^\dagger_j \\ &+ \tfrac{1}{2}K^k_{ij}\Delta^i\Delta^j\Delta^\dagger_k + \tfrac{1}{2}K^{ij}_k\Delta^\dagger_i\Delta^\dagger_j\Delta^k + \tfrac{1}{4}K^{kl}_{ij}\Delta^i\Delta^j\Delta^\dagger_k\Delta^\dagger_l \ , \end{aligned} \tag{7.8}$$

where

$$\begin{aligned}
\Delta^i\Delta^j &= -\theta\theta\psi^i\psi^j - \tfrac{i}{\sqrt{2}}\left(\psi^i\sigma^\mu\overline{\theta}\partial_\mu z^j + \psi^j\sigma^\mu\overline{\theta}\partial_\mu z^i\right) - \tfrac{1}{2}\theta\theta\overline{\theta\theta}\partial_\mu z^i\partial^\mu z^j \\
\Delta^i\Delta^\dagger_j &= \theta\sigma^\mu\overline{\theta}\ \psi^i\sigma_\mu\overline{\psi}_j \\
&\quad -\sqrt{2}\theta\theta\left(\overline{\theta\psi}_j f^i - \tfrac{i}{2}\psi^i\sigma^\mu\overline{\theta}\partial_\mu z^\dagger_j\right) - \sqrt{2}\overline{\theta\theta}\left(\theta\psi^i f^\dagger_j + \tfrac{i}{2}\theta\sigma^\mu\overline{\psi}_j\partial_\mu z^i\right) \\
&\quad +\theta\theta\overline{\theta\theta}\left(f^i f^\dagger_j + \tfrac{1}{2}\partial_\mu z^i\partial^\mu z^\dagger_j - \tfrac{i}{2}\psi^i\sigma^\mu\partial_\mu\overline{\psi}_j + \tfrac{i}{2}\partial_\mu\psi^i\sigma^\mu\overline{\psi}_j\right) \\
\Delta^i\Delta^j\Delta^\dagger_k &= -\sqrt{2}\theta\theta\,\overline{\theta\psi}_k\,\psi^i\psi^j \\
&\quad +\tfrac{i}{2}\theta\theta\overline{\theta\theta}\left(\psi^i\sigma^\mu\overline{\psi}_k\partial_\mu z^j + \psi^j\sigma^\mu\overline{\psi}_k\partial_\mu z^i - 2i\psi^i\psi^j f^\dagger_k\right) \\
\Delta^i\Delta^j\Delta^\dagger_k\Delta^\dagger_l &= \theta\theta\overline{\theta\theta}\psi^i\psi^j\,\overline{\psi}_k\overline{\psi}_l\ .
\end{aligned} \tag{7.9}$$

It is then easy to extract the D-term, i.e. the coefficient of $\theta\theta\overline{\theta\theta}$

$$\begin{aligned}
\int \mathrm{d}^2\theta\mathrm{d}^2\overline{\theta}\ K(\phi^i,\phi^\dagger_i) &= -\tfrac{1}{4}K_i\partial^2 z^i - \tfrac{1}{4}K^i\partial^2 z^\dagger_i - \tfrac{1}{4}K_{ij}\partial_\mu z^i\partial^\mu z^j + h.c. \\
&+ K^j_i\left(f^i f^\dagger_j + \tfrac{1}{2}\partial_\mu z^i\partial^\mu z^\dagger_j - \tfrac{i}{2}\psi^i\sigma^\mu\partial_\mu\overline{\psi}_j + \tfrac{i}{2}\partial_\mu\psi^i\sigma^\mu\overline{\psi}_j\right) \\
&+ \tfrac{i}{4}K^k_{ij}\left(\psi^i\sigma^\mu\overline{\psi}_k\partial_\mu z^j + \psi^j\sigma^\mu\overline{\psi}_k\partial_\mu z^i - 2i\psi^i\psi^j f^\dagger_k\right) + h.c. \\
&+ \tfrac{1}{4}K^{kl}_{ij}\psi^i\psi^j\,\overline{\psi}_k\overline{\psi}_l\ .
\end{aligned} \tag{7.10}$$

Next note that

$$\begin{aligned}
\partial_\mu\partial^\mu K(z^i,z^\dagger_j) &= K_i\partial^2 z^i + K^i\partial^2 z^\dagger_i + 2K^j_i\partial_\mu z^\dagger_j\partial^\mu z^i \\
&+ K_{ij}\partial_\mu z^i\partial^\mu z^j + K^{ij}\partial_\mu z^\dagger_i\partial^\mu z^\dagger_j
\end{aligned} \tag{7.11}$$

so that we can rewrite (7.10) as

$$\begin{aligned}
\int \mathrm{d}^2\theta\mathrm{d}^2\overline{\theta}\ K(\phi^i,\phi^\dagger_i) &= K^j_i\left(f^i f^\dagger_j + \partial_\mu z^i\partial^\mu z^\dagger_j - \tfrac{i}{2}\psi^i\sigma^\mu\partial_\mu\overline{\psi}_j + \tfrac{i}{2}\partial_\mu\psi^i\sigma^\mu\overline{\psi}_j\right) \\
&+ \tfrac{i}{4}K^k_{ij}\left(\psi^i\sigma^\mu\overline{\psi}_k\partial_\mu z^j + \psi^j\sigma^\mu\overline{\psi}_k\partial_\mu z^i - 2i\psi^i\psi^j f^\dagger_k\right) + h.c. \\
&+ \tfrac{1}{4}K^{kl}_{ij}\psi^i\psi^j\,\overline{\psi}_k\overline{\psi}_l - \tfrac{1}{4}\partial_\mu\partial^\mu K(z^i,z^\dagger_j)\ .
\end{aligned} \tag{7.12}$$

where the last term is a total derivative and hence can be dropped from the Lagrangian.

Note that after discarding this total derivative, (7.12) no longer contains the "purely holomorphic" terms $\sim K_{ij}$ or the "purely antiholomorphic" terms $\sim K^{ij}$.

Only the mixed terms with at least one upper and one lower index remain. This shows that the transformation

$$K(z,z^\dagger) \to K(z,z^\dagger) + g(z) + \overline{g}(z^\dagger) \tag{7.13}$$

does not affect the Lagrangian. Moreover, the metric of the kinetic terms for the complex scalars is

$$K_i^j = \frac{\partial^2}{\partial z^i \partial z_j^\dagger} K(z,z^\dagger) \ . \tag{7.14}$$

A metric like this obtained from a complex scalar function is called a Kähler metric, and the scalar function $K(z,z^\dagger)$ the Kähler potential. The metric is invariant under Kähler transformations (7.13) of this potential. Thus one is led to interpret the complex scalars z^i as (local) complex coordinates on a Kähler manifold, i.e. the target manifold of the sigma-model is Kähler. The Kähler invariance (7.13) actually generalises to the superfield level since

$$K(\phi,\phi^\dagger) \to K(\phi,\phi^\dagger) + g(\phi) + \overline{g}(\phi^\dagger) \tag{7.15}$$

does not affect the resulting action because $g(\phi)$ is again a chiral superfield and its $\theta\theta\overline{\theta}\overline{\theta}$ component is a total derivative, see (4.20), hence $\int \mathrm{d}^2\theta \mathrm{d}^2\overline{\theta}\ g(\phi) = \int \mathrm{d}^2\theta \mathrm{d}^2\overline{\theta}\ \overline{g}(\phi^\dagger) = 0$.

Once K_i^j is interpreted as a metric it is straightforward to compute the affine connection and curvature tensor. However, in Riemannian geometry, indices are lowered and rised by the metric and its inverse, while here we used upper and lower indices to denote derivatives w.r.t. z^i or $z_j^\dagger$. To avoid confusion, we temporarily switch conventions, replacing $z_j^\dagger \to z^{\overline{j}}$. Then $K_i^j \to K_{i\overline{j}}$ so that

$$K_{i\overline{j}} = K_{\overline{j}i} \quad , \quad K_{ij} = K_{\overline{i}\overline{j}} = 0 \tag{7.16}$$

and the inverse metric is $K^{i\overline{j}} = K^{\overline{j}i}$, $K^{ij} = K^{\overline{i}\overline{j}} = 0$. The affine connection is given as usual by $\Gamma^c_{ab} = \frac{1}{2} G^{cd}\left(\partial_a G_{bd} + \partial_b G_{ad} - \partial_d G_{ab}\right)$ which for the Kähler metric simplifies since $\frac{\partial}{\partial z^i} K_{j\overline{m}} = \frac{\partial}{\partial z^j} K_{i\overline{m}}$, etc. One finds

$$\Gamma^l_{ij} = K^{l\overline{m}} K_{ij\overline{m}} \quad , \quad \Gamma^{\overline{l}}_{\overline{i}\overline{j}} = K^{\overline{l}m} K_{\overline{i}\overline{j}m} \ , \tag{7.17}$$

all others with mixed indices like $\Gamma^l_{i\overline{j}}$ or $\Gamma^{\overline{l}}_{ij}$ vanish. The curvature tensor is given in general by

$$(R_{ab})^c{}_d = \partial_a \Gamma^c_{bd} - \partial_b \Gamma^c_{ad} + \Gamma^c_{af}\Gamma^f_{bd} - \Gamma^c_{bf}\Gamma^f_{ad} \ . \tag{7.18}$$

It is easy to see that in the Kähler case the only nonvanishing components are

$$(R_{\overline{k}i})^l{}_j = \partial_{\overline{k}} \Gamma^l_{ij} = K^{l\overline{p}}\left(K_{ij\overline{p}\overline{k}} - K_{ij\overline{m}} K^{\overline{m}n} K_{n\overline{p}\overline{k}}\right) \tag{7.19}$$

and $(R_{i\overline{k}})^l{}_j = -(R_{\overline{k}i})^l{}_j$, and similarly

$$(R_{i\overline{k}})^{\overline{l}}{}_{\overline{j}} = -(R_{\overline{k}i})^{\overline{l}}{}_{\overline{j}} = \partial_i \Gamma^{\overline{l}}_{\overline{kj}} = K^{\overline{l}p}\left(K_{ip\overline{kj}} - K_{ip\overline{m}} K^{\overline{m}n} K_{n\overline{kj}}\right) \ . \tag{7.20}$$

Reverting to our previous notation, we write

$$K_{i\overline{j}} \to K_i^j \ , \quad \Gamma^l_{ij} \to \Gamma^l_{ij} \ , \quad \Gamma^{\overline{l}}_{\overline{ij}} \to \Gamma^{ij}_l \ , \quad (R_{\overline{k}i})_{\overline{l}j} \to R^{kl}_{ij}, \tag{7.21}$$

i.e.

$$\begin{aligned} \Gamma^l_{ij} &= (K^{-1})^l_k K^k_{ij} \quad , \quad \Gamma^{ij}_l = (K^{-1})^k_l K^{ij}_k \ , \\ R^{kl}_{ij} &= K^{kl}_{ij} - K^m_{ij}(K^{-1})^n_m K^{kl}_n \ . \end{aligned} \tag{7.22}$$

This allows us to rewrite various terms in the Lagrangian in a simpler and more geometric form.

Define "Kähler covariant" derivatives of the fermions as

$$\begin{aligned} D_\mu \psi^i &= \partial_\mu \psi^i + \Gamma^i_{jk} \partial_\mu z^j \, \psi^k = \partial_\mu \psi^i + (K^{-1})^i_l \, K^l_{jk} \partial_\mu z^j \, \psi^k \\ D_\mu \overline{\psi}_j &= \partial_\mu \overline{\psi}_j + \Gamma^{ki}_j \partial_\mu z^\dagger_k \, \overline{\psi}_i = \partial_\mu \overline{\psi}_j + (K^{-1})^l_j \, K^{ki}_l \partial_\mu z^\dagger_k \, \overline{\psi}_i \ . \end{aligned} \tag{7.23}$$

The fermion bilinears in (7.12) then precisely are $\frac{i}{2} K^j_i D_\mu \psi^i \sigma^\mu \overline{\psi}_j + h.c.$. The four fermion term is $K^{kl}_{ij} \psi^i \psi^j \overline{\psi}_k \overline{\psi}_l$. The full curvature tensor will appear after we eliminate the auxiliary fields f^i. To do this, we add the two pieces (7.12) and (7.6) of the Lagrangian to see that the auxiliary field equations of motion are

$$f^i = (K^{-1})^i_j w^j - \frac{1}{2} \Gamma^i_{jk} \psi^j \psi^k \ . \tag{7.24}$$

Substituting back into the sum of (7.12) and (7.6) we finally get the Lagrangian

$$\begin{aligned} &\int \mathrm{d}^4 x \left[\int \, \mathrm{d}^2\theta \mathrm{d}^2\overline{\theta} \ K(\phi, \phi^\dagger) + \int \, \mathrm{d}^2\theta \ w(\phi) + \int \, \mathrm{d}^2\overline{\theta} \ [w(\phi)]^\dagger\right] \\ &= \int \mathrm{d}^4 x \Big[K^j_i \left(\partial_\mu z^i \partial^\mu z^\dagger_j + \tfrac{i}{2} D_\mu \psi^i \sigma^\mu \overline{\psi}_j - \tfrac{i}{2} \psi^i \sigma^\mu D_\mu \overline{\psi}_j\right) - (K^{-1})^i_j w_i w^j \\ &\quad -\tfrac{1}{2}\left(w_{ij} - \Gamma^k_{ij} w_k\right) \psi^i \psi^j - \tfrac{1}{2}\left(w^{ij} - \Gamma^{ij}_k w^k\right) \overline{\psi}_i \overline{\psi}_j + \tfrac{1}{4} R^{kl}_{ij} \psi^i \psi^j \overline{\psi}_k \overline{\psi}_l \Big] \ . \end{aligned} \tag{7.25}$$

7.2 Including gauge fields

The inclusion of gauge fields changes two things. First, the kinetic term $K(\phi, \phi^\dagger)$ has to be modified so that, among others, all derivatives ∂_μ are turned into gauge covariant derivatives as we did in section 4 when we replaced $\phi^\dagger \phi$ by $\phi^\dagger e^{2gV} \phi$.

Second, one has to add kinetic terms for the gauge multiplet V. In the spirit of the σ-model, one will allow a susy Lagrangian leading to terms of the form $f_{ab}(z)F^a_{\mu\nu}F^{b\mu\nu}$ etc.

Let's discuss the matter Lagrangian first. Since

$$\phi \to e^{i\Lambda}\phi\ , \quad \phi^\dagger \to \phi^\dagger e^{-i\Lambda^\dagger}\ , \quad e^{2gV} \to e^{i\Lambda^\dagger}e^{2gV}e^{-i\Lambda} \tag{7.26}$$

one sees that

$$\phi^\dagger e^{2gV} \to \phi^\dagger e^{2gV}e^{-i\Lambda}\ . \tag{7.27}$$

Then the combination $\left(\phi^\dagger e^{2gV}\right)_i \phi^i$ is gauge invariant and the same is true for any real (globally) G-invariant function $K(\phi^i, \phi^\dagger_i)$ if the argument $\phi^\dagger_i$ is replaced by $\left(\phi^\dagger e^{2gV}\right)_i$. We conclude that if $w(\phi^i)$ is a G-invariant function of the ϕ^i, i.e. if

$$w_i (T^a)^i{}_j \phi^j = 0 \quad , \quad a = 1, \ldots \dim G \tag{7.28}$$

then

$$\mathcal{L}_{\rm matter} = \int \mathrm{d}^2\theta \mathrm{d}^2\overline{\theta}\ K\left(\phi^i, \left(\phi^\dagger e^{2gV}\right)_i\right) + \int \mathrm{d}^2\theta\ w(\phi^i) + \int \mathrm{d}^2\overline{\theta}\ [w(\phi^i)]^\dagger \tag{7.29}$$

is supersymmetric and gauge invariant.

To discuss the generalisation of the gauge kinetic Lagrangian (5.17), reall that W_α is defined by (5.3) with $V \to 2gV$ and in WZ gauge it reduces to (5.10) times $2g$. Note that any power of W never contains more than two derivatives, so we could consider a susy Lagrangian of the form $\int \mathrm{d}^2\theta\ H(\phi^i, W_\alpha)$ with an arbitrary G-invariant function H. We will be slightly less general and take

$$\mathcal{L}_{\rm gauge} = \frac{1}{16g^2}\int \mathrm{d}^2\theta\ f_{ab}(\phi^i) W^{a\alpha}W^b_\alpha + h.c. \tag{7.30}$$

with $f_{ab} = f_{ba}$ transforming under G as the symmetric product of the adjoint representation with itself. To get back the standard Lagrangian (5.17) one only needs to take $\frac{1}{g^2}f_{ab} = \frac{\tau}{4\pi i}\mathrm{Tr}\, T^a T^b$. Expanding (7.30) in components is straightforward and yields

$$\begin{aligned}
\mathcal{L}_{\rm gauge} \quad = \quad & \mathrm{Re} f_{ab}(z)\left(-\tfrac{1}{4}F^a_{\mu\nu}F^{b\mu\nu} - i\lambda^a\sigma^\mu D_\mu\overline{\lambda}^b + \tfrac{1}{2}D^a D^b\right) - \tfrac{1}{4}\mathrm{Im} f_{ab}(z) F^a_{\mu\nu}\widetilde{F}^{b\mu\nu} \\
+ \quad & \tfrac{1}{4} f_{ab,i}(z)\left(\sqrt{2}i\psi^i\lambda^a D^b - \sqrt{2}\lambda^a\sigma^{\mu\nu}\psi^i F^b_{\mu\nu} + \lambda^a\lambda^b f^i\right) + h.c. \\
+ \quad & \tfrac{1}{8} f_{ab,ij}(z)\lambda^a\lambda^b\psi^i\psi^j + h.c.
\end{aligned} \tag{7.31}$$

where $F_{\mu\nu}$ and $D_\mu\lambda$ were defined in (5.15) and $f_{ab,i} = \frac{\partial}{\partial z^i} f_{ab}(z)$ etc.

To obtain the component expansion of the matter Lagrangian (7.29) is a bit lengthy. The computation parallels the one leading to (7.12) but paying attention to the gauge field terms. The result can be read from (7.12) by gauge covariantising all derivatives and adding (7.6). Furthermore, it is clear that one also obtains the Yukawa interactions that already appeared in (5.27) with the Kähler metric appropriately inserted. Note also that the term $gz^\dagger Dz$ now is replaced by $gz_i^\dagger DK^i$. Taking all this into account it is easy to see that one obtains

$$\begin{aligned}
\mathcal{L}_{\text{matter}} \;=\;& K_i^j \left[f^i f_j^\dagger + (D_\mu z)^i (D^\mu z)_j^\dagger - \tfrac{i}{2}\psi^i \sigma^\mu \widetilde{D}_\mu \overline{\psi}_j + \tfrac{i}{2}\widetilde{D}_\mu \psi^i \sigma^\mu \overline{\psi}_j \right] \\
&+ \tfrac{1}{2} K_{ij}^k \,\psi^i \psi^j f_k^\dagger + h.c. + \tfrac{1}{4} K_{ij}^{kl}\, \psi^i \psi^j\, \overline{\psi}_k \overline{\psi}_l \\
&- \left(w_i f^i + \tfrac{1}{2} w_{ij} \psi^i \psi^j \right) + h.c. \\
&+ i\sqrt{2} g K_j^i\, z_i^\dagger \lambda \psi^j - i\sqrt{2} g K_j^i\, \overline{\psi}_i \overline{\lambda} z^j + g z_i^\dagger D K^i \;,
\end{aligned} \tag{7.32}$$

where as before gauge indices have been suppressed, e.g. $\overline{\psi}_i \overline{\lambda} z^j \equiv \overline{\psi}_i T_R^a z^j \overline{\lambda}^a \equiv (\overline{\psi}_i)_M (T_R^a)^M_N (z^j)^N \overline{\lambda}^a$ where $(T_R^a)^M_N$ are the matrices of the representation carried by the matter fields $(z^j)^N$ and $(\psi^i)^N$. The derivatives $\widetilde{D}_\mu$ acting on the fermions are gauge and Kähler covariant, i.e.

$$\begin{aligned}
\widetilde{D}_\mu \psi^i &= \partial_\mu \psi^i - i g v_\mu^a T_R^a \psi^i + \Gamma^i_{jk} \partial_\mu z^j\, \psi^k \\
\widetilde{D}_\mu \overline{\psi}_j &= \partial_\mu \overline{\psi}_j - i g v_\mu^a T_R^a \overline{\psi}_j + \Gamma^{ki}_j \partial_\mu z_k^\dagger\, \overline{\psi}_i \;.
\end{aligned} \tag{7.33}$$

The full Lagrangian is given by $\mathcal{L} = \mathcal{L}_{\text{gauge}} + \mathcal{L}_{\text{matter}}$. The auxiliary field equations of motion are

$$\begin{aligned}
f^i &= (K^{-1})^i_j \left(w^j - \tfrac{1}{2} K^j_{kl}\, \psi^k \psi^l - \tfrac{1}{4} (f_{ab,j})^\dagger \overline{\lambda}^a \overline{\lambda}^b \right) \\
D^a &= -(\mathrm{Re} f)^{-1}_{ab} \left(g z_i^\dagger T^b K^i + \tfrac{i}{2\sqrt{2}} f_{bc,i} \psi^i \lambda^c - \tfrac{i}{2\sqrt{2}} (f_{bc,i})^\dagger \overline{\psi}_i \overline{\lambda}^c \right) \;.
\end{aligned} \tag{7.34}$$

It is straightforward to substitute this into the Lagrangian $\mathcal{L}$ and we will not write the result explicitly. Let us only note that the scalar potential is given by

$$V(z, z^\dagger) = (K^{-1})^i_j w_i w^j + \frac{g^2}{2} (\mathrm{Re} f)^{-1}_{ab} (z_i^\dagger T^a K^i)(z_j^\dagger T^b K^j) \;. \tag{7.35}$$

Chapter 8

$N = 2$ susy gauge theory

The $N = 2$ multiplets with helicities not exceeding one are the massless $N = 2$ vector multiplet and the hypermultiplet. The former contains an $N = 1$ vector multiplet and an $N = 1$ chiral multiplet, alltogether a gauge boson, two Weyl fermions and a complex scalar, while the hypermultiplet contains two $N = 1$ chiral multiplets. The $N = 2$ vector multiplet is necessarily massless while the hypermultiplet can be massless or be a short (BPS) massive multiplet. Here we will concentrate on the $N = 2$ vector multiplet.

8.1 $N = 2$ super Yang-Mills

Given the decomposition of the $N = 2$ vector multiplet into $N = 1$ multiplets, we start with a Lagrangian being the sum of the $N = 1$ gauge and matter Lagrangians (5.17) and (5.25). At present, however, all fields are in the same $N = 2$ multiplet and hence must be in the same representation of the gauge group, namely the adjoint representation. The $N = 1$ matter Lagrangian (5.25) then becomes, after rescaling $V \to 2gV$,

$$\begin{aligned} \mathcal{L}^{N=1}_{\rm matter} = \int \mathrm{d}^2\theta \mathrm{d}^2\overline{\theta}\ \mathrm{Tr}\, \phi^\dagger e^{2gV}\phi &= \mathrm{Tr}\Big[(D_\mu z)^\dagger D^\mu z - i\psi\sigma^\mu D_\mu\overline{\psi} + f^\dagger f \\ &+ i\sqrt{2}gz^\dagger\{\lambda,\psi\} - i\sqrt{2}g\{\overline{\psi},\overline{\lambda}\}z + gD[z,z^\dagger]\Big] \end{aligned} \tag{8.1}$$

where now

$$z = z^a T^a\ ,\quad \psi = \psi^a T^a\ ,\quad f = f^a T^a\ ,\quad a = 1,\ldots \mathrm{dim}G \tag{8.2}$$

in addition to $\lambda = \lambda^a T^a$, $D = D^a T^a$, $v_\mu = v^a_\mu T^a$. The commutators or anti-commutators arise since the generators in the adjoint representation are given by

$$(T^a_{\rm ad})_{bc} = -i f_{abc} \tag{8.3}$$

and we normalise the generators by

$$\mathrm{Tr}\, T^a T^b = \delta^{ab} \tag{8.4}$$

so that

$$\begin{aligned} z^\dagger \lambda \psi \quad &\rightarrow \quad z^\dagger_b \lambda^a \, (T^a_{\rm ad})_{bc}\, \psi^c = -i z^\dagger_b \lambda^a f_{abc} \psi^c = i z^\dagger_b f_{bac} \lambda^a \psi^c \\ &= \quad z^\dagger_b \lambda^a \psi^c \mathrm{Tr}\, T^b [T^a, T^c] = \mathrm{Tr}\, z^\dagger \{\lambda, \psi\} \end{aligned} \tag{8.5}$$

and

$$z^\dagger D z \rightarrow z^\dagger_b D^a \, (T^a_{\rm ad})_{bc}\, z^c = -i f_{abc} z^\dagger_b D^a z^c = -\mathrm{Tr}\, D[z^\dagger, z] = \mathrm{Tr}\, D[z, z^\dagger] \; . \tag{8.6}$$

We now add (8.1) to the $N = 1$ gauge lagrangian $\mathcal{L}^{N=1}_{\rm gauge}$ (5.17) and obtain

$$\begin{aligned} \mathcal{L}^{N=2}_{\rm YM} \quad &= \quad \tfrac{1}{32\pi} \mathrm{Im}\, (\tau \textstyle\int \mathrm{d}^2\theta \; \mathrm{Tr}\, W^\alpha W_\alpha) + \int \mathrm{d}^2\theta \mathrm{d}^2\overline{\theta} \; \mathrm{Tr}\, \phi^\dagger e^{2gV} \phi \\ &= \quad \mathrm{Tr} \Big(-\tfrac{1}{4} F_{\mu\nu} F^{\mu\nu} - i\lambda\sigma^\mu D_\mu \overline{\lambda} - i\psi\sigma^\mu D_\mu \overline{\psi} + (D_\mu z)^\dagger D^\mu z \\ &\quad + \tfrac{\Theta}{32\pi^2} g^2 \, \mathrm{Tr} F_{\mu\nu} \widetilde{F}^{\mu\nu} + \tfrac{1}{2} D^2 + f^\dagger f \\ &\quad + i\sqrt{2} g z^\dagger \{\lambda, \psi\} - i\sqrt{2} g \{\overline{\psi}, \overline{\lambda}\} z + g D[z, z^\dagger] \Big) \; . \end{aligned} \tag{8.7}$$

A necessary and sufficient condition for $N = 2$ susy is the existence of an $\mathrm{SU}(2)_R$ symmetry that rotates the two supersymmetry generators Q^1_α and Q^2_α into each other. As follows from the construction of the supermultiplet in section 2, the same symmetry must act between the two fermionic fields λ and ψ. Now the relative coefficients of $\mathcal{L}^{N=1}_{\rm gauge}$ and $\mathcal{L}^{N=1}_{\rm matter}$ in (8.7) have been chosen precisely in such a way to have this $\mathrm{SU}(2)_R$ symmetry: the λ and ψ kinetic terms have the same coefficient, and the Yukawa couplings $z^\dagger\{\lambda, \psi\}$ and $\{\overline{\psi}, \overline{\lambda}\} z$ also exhibit this symmetry. The Lagrangian (8.7) is indeed $N = 2$ supersymmetric.

Note that we have not added a superpotential. Such a term (unless linear in ϕ) would break the $\mathrm{SU}(2)_R$ invariance and not lead to an $N = 2$ theory.

The auxiliary field equations of motion are simply

$$\begin{aligned} f^a \quad &= \quad 0 \\ D^a \quad &= \quad -g\, [z, z^\dagger]^a \end{aligned} \tag{8.8}$$

leading to a scalar potential

$$V(z, z^\dagger) = \frac{1}{2} g^2 \; \mathrm{Tr} \, \Big([z, z^\dagger] \Big)^2 \; . \tag{8.9}$$

This scalar potential is fixed and a consequence solely of the auxiliary D-field of the $N = 1$ gauge multiplet.

8.2 Effective $N = 2$ gauge theories

As for the non-linear σ-model, if one considers effective theories, disregarding renormalisability, one may allow more general gauge and matter kinetic terms and start with an appropriate sum of (7.29) (with $w(\phi^i) = 0$) and (7.30). It is clear however that the functions f_{ab} cannot be independent from the Kähler potential K. Indeed, the $\mathrm{SU}(2)_R$ symmetry equates $\mathrm{Re} f_{ab}$ with the Kähler metric K_a^b. It turns out that this requires the following identification

$$\begin{aligned} \tfrac{16\pi}{(2g)^2}\, f_{ab}(z) &= -i\tfrac{\partial^2}{\partial z^a \partial z^b}\mathcal{F}(z) \equiv -i\mathcal{F}_{ab}(z) \\ \tfrac{16\pi}{(2g)^2}\, K(z,z^\dagger) &= -\tfrac{i}{2}\, z_a^\dagger \tfrac{\partial}{\partial z^a}\mathcal{F}(z) + h.c. \equiv -\tfrac{i}{2}\, z_a^\dagger \mathcal{F}_a(z) + \tfrac{i}{2}\, [\mathcal{F}_a(z)]^\dagger\, z_a \end{aligned} \tag{8.10}$$

where the holomorphic function $\mathcal{F}(z)$ is called the $N = 2$ prepotential. We have pulled out a factor $\frac{16\pi}{(2g)^2}$ for later convenience. Also, we again absorb the factor $2g$ into the normalisation of the field. This makes sense since $\mathrm{Im}\mathcal{F}_{ab}$ will play the role of an effective generalised coupling. Hence we set

$$2g = 1. \tag{8.11}$$

Then the full general $N = 2$ Lagrangian is

$$\begin{aligned} \mathcal{L}_{\mathrm{eff}}^{N=2} &= \left[\tfrac{1}{64\pi i} \int \mathrm{d}^2\theta\ \mathcal{F}_{ab}(\phi) W^{a\alpha} W_\alpha^b + \tfrac{1}{32\pi i}\int \mathrm{d}^2\theta \mathrm{d}^2\overline{\theta}\ \left(\phi^\dagger e^V\right)^a \mathcal{F}_a(\phi)\right] + h.c. \\ &= \tfrac{1}{16\pi}\mathrm{Im}\left[\tfrac{1}{2}\int \mathrm{d}^2\theta\ \mathcal{F}_{ab}(\phi) W^{a\alpha} W_\alpha^b + \int \mathrm{d}^2\theta \mathrm{d}^2\overline{\theta}\ \left(\phi^\dagger e^V\right)^a \mathcal{F}_a(\phi)\right]\ . \end{aligned} \tag{8.12}$$

Note that with the Kähler potential K given by (8.10), the Kähler metric is proportional to $\mathrm{Im}\mathcal{F}_{ab}$ as required by $\mathrm{SU}(2)_R$:

$$K_a^b = \frac{1}{16\pi}\mathrm{Im}\mathcal{F}_{ab} = \frac{1}{32\pi i}\left(\mathcal{F}_{ab} - \mathcal{F}_{ab}^\dagger\right)\ . \tag{8.13}$$

The component expansion follows from the results of the previous section on the non-linear σ-model, using the identifications (8.10) and (8.13), and taking vanishing superpotential $w(\phi)$. In particular, the scalar potential is given by (cf (7.35))

$$V(z,z^\dagger) = -\frac{1}{2\pi}(\mathrm{Im}\mathcal{F})^{-1}_{ab}\,[z^\dagger, \mathcal{F}_c(z)T^c]^a\,[z^\dagger, \mathcal{F}_d(z)T^d]^b\ . \tag{8.14}$$

Let us insist that the full effective $N = 2$ action written in (8.12) is determined by a single holomorphic function $\mathcal{F}(z)$. Holomorphicity will turn out to be a very strong requirement. Finally note that $\mathcal{F}(z) = \frac{1}{2}\tau\,\mathrm{Tr}\, z^2$ gives back the standard Yang-Mills Lagrangian (8.7).

Chapter 9

Seiberg-Witten duality in $N = 2$ gauge theory

In this section, I will discuss how electric-magnetic duality is realised in an effective low-energy $N = 2$ gauge theory. This was pioneered by Seiberg and Witten in 1994 [10] who considered the simplest case of pure $N = 2$ supersymmetric SU(2) Yang-Mills theory. This work was then generalized to other gauge groups and to theories including extra matter fields (susy QCD). In the mean time, it became increasingly clear that dualities in string theories play an even more fascinating role (as is discussed by others at this school). Here I focus on the simplest SU(2) case which most clearly examplifies the beauty of duality. This section is based on an earlier introduction into the subject by the present author [11] where further references can be found.

The idea of duality probably goes back to Dirac who observed that the source-free Maxwell equations are symmetric under the exchange of the electric and magnetic fields. More precisely, the symmetry is $E \to B,\ B \to -E$, or $F_{\mu\nu} \to \tilde{F}_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu}{}^{\rho\sigma}F_{\rho\sigma}$. To maintain this symmetry in the presence of sources, Dirac introduced, somewhat ad hoc, magnetic monopoles with magnetic charges q_m in addition to the electric charges q_e, and showed that consistency of the quantum theory requires a charge quantization condition $q_m q_e = 2\pi n$ with integer n. Hence the minimal charges obey $q_m = \frac{2\pi}{q_e}$. Duality exchanges q_e and q_m, i.e. q_e and $\frac{2\pi}{q_e}$. Now recall that the electric charge q_e also is the coupling constant. So duality exchanges the coupling constant with its inverse (up to the factor of 2π), hence exchanging strong and weak coupling. This is the reason why we are so much interested in duality: the hope is to learn about strong-coupling physics from the weak-coupling physics of a dual formulation of the theory. Of course, in classical Maxwell theory we know all we may want to know, but this is no longer true in quantum electrodynamics.

Actually, quantum electrodynamics is not a good candidate for exhibiting a duality symmetry since there are no magnetic monopoles, but the latter naturally appear in spontaneously broken non-abelian gauge theories. Unfortunately, electric-magnetic duality in its simplest form cannot be a symmetry of the quantum theory due to the running of the coupling constant (among other reasons). Indeed, if duality exchanges $\alpha(\Lambda) \leftrightarrow \frac{1}{\alpha(\Lambda)}$ (where $\alpha(\Lambda) = \frac{e^2(\Lambda)}{4\pi}$) at some scale Λ, in general this won't be true at another scale. This argument is avoided if the coupling does not run, i.e. if the β-function vanishes as is the case in certain ($N = 4$) supersymmetric extensions of the Yang-Mills theory. This and other reasons led Montonen and Olive to conjecture that duality might be an exact symmetry of $N = 4$ susy Yang-Mills theory. The Seiber-Witten duality concerns a different type of theory: it deals with an $N = 2$ susy low-energy *effective* action and duality exchanges the effective coupling $\alpha_{\rm eff}(\Lambda)$ with a dual coupling $\alpha_{\rm eff}^D(\Lambda_D) \sim \frac{1}{\alpha_{\rm eff}(\Lambda)}$ at a dual scale Λ_D. The dependence of this dual scale Λ_D on the original scale Λ precisely takes into account the running of the coupling. Let me insist that the Seiberg-Witten duality is an *exact* symmetry of the abelian low-energy *effective* theory, not of the microscopic $SU(2)$ theory. This is different from the Montonen-Olive conjecture about an exact duality symmetry of a microscopic gauge theory.

A somewhat similar duality symmetry appears in the two-dimensional Ising model where it exchanges the temperature with a dual temperature, thereby exchanging high and low temperature analogous to strong and weak coupling. For the Ising model, the sole existence of the duality symmetry led to the exact determination of the critical temperature as the self-dual point, well prior to the exact solution by Onsager. One may view the existence of this self-dual point as the requirement that the dual high and low temperature regimes can be consistently "glued" together. Similarly, in the Seiberg-Witten theory, as will be explained below, duality allows us to obtain the full effective action for the light fields at any coupling (the analogue of the Ising free energy at any temperature) from knowledge of its weak-coupling limit and the behaviour at certain strong-coupling "singularities", together with a holomorphicity requirement that tells us how to patch together the different limiting regimes.

9.1 Low-energy effective action of $N = 2$ $SU(2)$ YM theory

Following Seiberg and Witten we want to study and determine the low-energy effective action of the $N = 2$ susy Yang-Mills theory with gauge group SU(2). The latter theory is the microscopic theory which controls the high-energy behaviour.

It was discussed in section 6 and its Lagrangian is given by (8.7). This theory is renormalisable and well-known to be asymptotically free. The low-energy effective action will turn out to be quite different.

9.1.1 Low-energy effective actions

There are two types of effective actions. One is the standard generating functional $\Gamma[\varphi]$ of one-particle irreducible Feynman diagrams (vertex functions). It is obtained from the standard renormalised generating functional $W[\varphi]$ of connected diagrams by a Legendre transformation. Momentum integrations in loop-diagrams are from zero up to a UV-cutoff which is taken to infinity after renormalisation. $\Gamma[\varphi] \equiv \Gamma[\mu, \varphi]$ also depends on the scale μ used to define the renormalized vertex functions.

A quite different object is the Wilsonian effective action $S_{\rm W}[\mu, \varphi]$. It is defined as $\Gamma[\mu, \varphi]$, except that all loop-momenta are only integrated down to μ which serves as an infra-red cutoff. In theories with massive particles only, there is no big difference between $S_{\rm W}[\mu, \varphi]$ and $\Gamma[\mu, \varphi]$ (as long as μ is less than the smallest mass). When massless particles are present, as is the case for gauge theories, the situation is different. In particular, in supersymmetric gauge theories there is the so-called Konishi anomaly which can be viewed as an IR-effect. Although $S_{\rm W}[\mu, \varphi]$ depends holomorphically on μ, this is not the case for $\Gamma[\mu, \varphi]$ due to this anomaly.

9.1.2 The SU(2) case, moduli space

Following Seiberg and Witten, we want to determine the Wilsonian effective action in the case where the microscopic theory is the SU(2), $N = 2$ super Yang-Mills theory. As explained above, classically this theory has a scalar potential $V(z) = \frac{1}{2}g^2 \mathrm{tr}\ ([z^\dagger, z])^2$ as given in (8.9). Unbroken susy requires that $V(z) = 0$ in the vacuum, but this still leaves the possibilities of non-vanishing z with $[z^\dagger, z] = 0$. We are interested in determining the gauge inequivalent vacua. A general z is of the form $z(x) = \frac{1}{2}\sum_{j=1}^{3} (a_j(x) + ib_j(x))\, \sigma_j$ with real fields $a_j(x)$ and $b_j(x)$ (where I assume that not all three a_j vanish, otherwise exchange the roles of the a_j's and b_j's in the sequel). By a SU(2) gauge transformation one can always arrange $a_1(x) = a_2(x) = 0$. Then $[z, z^\dagger] = 0$ implies $b_1(x) = b_2(x) = 0$ and hence, with $a = a_3 + ib_3$, one has $z = \frac{1}{2}a\sigma_3$. Obviously, in the vacuum a must be a constant. Gauge transformation from the Weyl group (i.e. rotations by π around the 1- or 2-axis of SU(2)) can still change $a \to -a$, so a and $-a$ are gauge equivalent, too. The gauge invariant quantity describing inequivalent vacua is $\frac{1}{2}a^2$, or tr z^2, which is the same, semiclassically. When quantum fluctuations are important this is no longer so. In the sequel, we will use the following definitions for a and u:

$$u = \langle \mathrm{tr}\ z^2 \rangle \quad , \quad \langle z \rangle = \frac{1}{2} a \sigma_3 \ . \tag{9.1}$$

The complex parameter u labels gauge inequivalent vacua. The manifold of gauge inequivalent vacua is called the moduli space $\mathcal{M}$ of the theory. Hence u is a coordinate on $\mathcal{M}$, and $\mathcal{M}$ is essentially the complex u-plane. We will see in the sequel that $\mathcal{M}$ has certain singularities, and the knowledge of the behaviour of the theory near the singularities will eventually allow the determination of the effective action S_{W}.

Clearly, for non-vanishing $\langle z \rangle$, the SU(2) gauge symmetry is broken by the Higgs mechanism, since the z-kinetic term $|D_\mu z|^2$ generates masses for the gauge fields. With the above conventions, $v_\mu^b,\ b = 1, 2$ become massive with masses given by $\frac{1}{2} m^2 = g^2 |a|^2$, i.e $m = \sqrt{2} g |a|$. Similarly due to the ϕ, λ, ψ interaction terms, $\psi^b, \lambda^b,\ b = 1, 2$ become massive with the same mass as the v_μ^b, as required by supersymmetry. Obviously, v_μ^3, ψ^3 and λ^3, as well as the mode of ϕ describing the flucuation of ϕ in the σ_3-direction, remain massless. These massless modes are described by a Wilsonian low-energy effective action which has to be $N = 2$ supersymmetry invariant, since, although the gauge symmetry is broken, SU(2) $\to$ U(1), the $N = 2$ susy remains unbroken. Thus it must be of the general form (8.12) where the indices a, b now take only a single value ($a, b = 3$) and will be suppressed since the gauge group is U(1). Also, in an abelian theory there is no self-coupling of the gauge boson and the same arguments extend to all members of the $N = 2$ susy multiplet: they do not carry electric charge. Thus for a U(1)-gauge theory, from (8.12) we simply get

$$\frac{1}{16\pi} \mathrm{Im} \int \mathrm{d}^4 x \left[\frac{1}{2} \int \mathrm{d}^2\theta\, \mathcal{F}''(\phi) W^\alpha W_\alpha + \int \mathrm{d}^2\theta\, \mathrm{d}^2\bar{\theta}\, \phi^\dagger \mathcal{F}'(\phi) \right] \ . \tag{9.2}$$

9.1.3 Metric on moduli space

As shown in (8.13), the Kähler metric of the present σ-model is given by $K_{z\bar{z}} = \frac{1}{16\pi} \mathrm{Im}\, \mathcal{F}''(z)$. By the same token this defines the metric in the space of (inequivalent) vacuum configurations, i.e. the metric on moduli space as ($\bar{a}$ denotes the complex conjugate of a)

$$\mathrm{d}s^2 = \mathrm{Im}\, \mathcal{F}''(a) \mathrm{d}a \mathrm{d}\bar{a} = \mathrm{Im}\, \tau(a) \mathrm{d}a \mathrm{d}\bar{a} \tag{9.3}$$

where $\tau(a) = \mathcal{F}''(a)$ is the effective (complexified) coupling constant according to the remark after eq. (7.30). The σ-model metric $K_{z\bar{z}}$ has been replaced on the moduli space $\mathcal{M}$ by (16π times) its expectation value in the vacuum corresponding to the given point on $\mathcal{M}$, i.e. by $\mathrm{Im}\, \mathcal{F}''(a)$.

The question now is whether the description of the effective action in terms of the fields ϕ, W and the function $\mathcal{F}$ is appropriate for all vacua, i.e. for all value of u, i.e. on all of moduli space. In particular the kinetic terms or what is the same, the metric on moduli space should be positive definite, translating into $\mathrm{Im}\,\tau(a) > 0$. However, a simple argument shows that this cannot be the case: since $\mathcal{F}(a)$ is holomorphic, $\mathrm{Im}\,\tau(a) = \mathrm{Im}\,\frac{\partial^2 \mathcal{F}(a)}{\partial a^2}$ is a harmonic function and as such it cannot have a minimum, and hence (on the compactified complex plane) it cannot obey $\mathrm{Im}\,\tau(a) > 0$ everywhere (unless it is a constant as in the classical case). The way out is to allow for different local descriptions: the coordinates $a, \bar{a}$ and the function $\mathcal{F}(a)$ are appropriate only in a certain region of $\mathcal{M}$. When a singular point with $\mathrm{Im}\,\tau(a) \to 0$ is approached one has to use a different set of coordinates $\hat{a}$ in which $\mathrm{Im}\,\hat{\tau}(\hat{a})$ is non-singular (and non-vanishing). This is possible provided the singularity of the metric is only a coordinate singularity, i.e. the kinetic terms of the effective action are not intrinsically singular, which will be the case.

9.1.4 Asymptotic freedom and the one-loop formula

Classically the function $\mathcal{F}(z)$ is given by $\frac{1}{2}\tau_{\text{class}} z^2$. The one-loop contribution has been determined by Seiberg. The combined tree-level and one-loop result is

$$\mathcal{F}_{\text{pert}}(z) = \frac{i}{2\pi} z^2 \ln \frac{z^2}{\Lambda^2} \ . \tag{9.4}$$

Here Λ^2 is some combination of μ^2 and numerical factors chosen so as to fix the normalisation of $\mathcal{F}_{\text{pert}}$. Note that due to non-renormalisation theorems for $N=2$ susy there are no corrections from two or more loops to the Wilsonian effective action S_{W} and (9.4) is the full perturbative result. There are however non-perturbative corrections that will be determined below.

For very large a the dominant contribution when computing S_{W} from the microscopic SU(2) gauge theory comes from regions of large momenta ($p \sim a$) where the microscopic theory is asymptotically free. Thus, as $a \to \infty$ the effective coupling constant goes to zero, and the perturbative expression (9.4) for $\mathcal{F}$ becomes an excellent approximation. Also $u \sim \frac{1}{2}a^2$ in this limit.[1] Thus

$$\begin{aligned} \mathcal{F}(a) &\sim \tfrac{i}{2\pi} a^2 \ln \tfrac{a^2}{\Lambda^2} \\ \tau(a) &\sim \tfrac{i}{\pi}\left(\ln \tfrac{a^2}{\Lambda^2} + 3\right) \quad \text{as } u \to \infty \ . \end{aligned} \tag{9.5}$$

[1] One can check from the explicit solution in section 6 that one indeed has $\frac{1}{2}a^2 - u = \mathcal{O}(1/u)$ as $u \to \infty$.

Note that due to the logarithm appearing at one-loop, $\tau(a)$ is a multi-valued function of $a^2 \sim 2u$. Its imaginary part, however, $\mathrm{Im}\,\tau(a) \sim \frac{1}{\pi}\ln\frac{|a|^2}{\Lambda^2}$ is single-valued and positive (for $a^2 \to \infty$).

9.2 Duality

As already noted, a and $\bar{a}$ do provide local coordinates on the moduli space $\mathcal{M}$ for the region of large u. This means that in this region ϕ and W^α are appropriate fields to describe the low-energy effective action. As also noted, this description cannot be valid globally, since $\mathrm{Im}\,\mathcal{F}''(a)$, being a harmonic function, must vanish somewhere, unless it is a constant - which it is not. Duality will provide a different set of (dual) fields ϕ_{D} and W^α_{D} that provide an appropriate description for a different region of the moduli space.

9.2.1 Duality transformation

Define a dual field ϕ_D and a dual function $\mathcal{F}_D(\phi_D)$ by

$$\phi_D = \mathcal{F}'(\phi) \quad , \quad \mathcal{F}'_D(\phi_D) = -\phi \ . \tag{9.6}$$

These duality transformations simply constitute a Legendre transformation $\mathcal{F}_D(\phi_D) = \mathcal{F}(\phi) - \phi\phi_D$. Using these relations, the second term in the ϕ kinetic term of the action can be written as

$$\begin{aligned} \mathrm{Im}\int \mathrm{d}^4x\,\mathrm{d}^2\theta\,\mathrm{d}^2\bar{\theta}\,\phi^+\mathcal{F}'(\phi) &= \mathrm{Im}\int \mathrm{d}^4x\,\mathrm{d}^2\theta\,\mathrm{d}^2\bar{\theta}\,\left(-\mathcal{F}'_D(\phi_D)\right)^+\phi_D \\ &= \mathrm{Im}\int \mathrm{d}^4x\,\mathrm{d}^2\theta\,\mathrm{d}^2\bar{\theta}\,\phi_D^+\mathcal{F}'_D(\phi_D)\ . \end{aligned} \tag{9.7}$$

We see that this second term in the effective action is invariant under the duality transformation.

Next, consider the $\mathcal{F}''(\phi)W^\alpha W_\alpha$-term in the effective action (9.2). While the duality transformation on ϕ is local, this will not be the case for the transformation of W^α. Recall that W contains the U(1) field strength $F_{\mu\nu}$. This $F_{\mu\nu}$ is not arbitrary but of the form $\partial_\mu v_\nu - \partial_\nu v_\mu$ for some v_μ. This can be translated into the Bianchi identity $\frac{1}{2}\epsilon^{\mu\nu\rho\sigma}\partial_\nu F_{\rho\sigma} \equiv \partial_\nu \tilde{F}^{\mu\nu} = 0$. The corresponding constraint in superspace is $\mathrm{Im}\,(D_\alpha W^\alpha) = 0$. In the functional integral one has the choice of integrating over V only, or over W^α and imposing the constraint $\mathrm{Im}\,(D_\alpha W^\alpha) = 0$

by a real Lagrange multiplier superfield which we call V_D:

$$\begin{aligned}
&\int \mathcal{D}V \exp\left[\tfrac{i}{32\pi}\mathrm{Im}\int \mathrm{d}^4x\,\mathrm{d}^2\theta\,\mathcal{F}''(\phi)W^\alpha W_\alpha\right]\\
&\simeq \int \mathcal{D}W\mathcal{D}V_D \exp\left[\tfrac{i}{32\pi}\mathrm{Im}\int \mathrm{d}^4x\left(\int \mathrm{d}^2\theta\,\mathcal{F}''(\phi)W^\alpha W_\alpha\right.\right.\\
&\qquad\qquad\qquad\qquad\left.\left.+\tfrac{1}{2}\int \mathrm{d}^2\theta\,\mathrm{d}^2\bar\theta\,V_D D_\alpha W^\alpha\right)\right]
\end{aligned}\tag{9.8}$$

Observe that

$$\begin{aligned}
\int \mathrm{d}^2\theta\,\mathrm{d}^2\bar\theta\,V_D D_\alpha W^\alpha &= -\int \mathrm{d}^2\theta\,\mathrm{d}^2\bar\theta\,D_\alpha V_D W^\alpha = +\int \mathrm{d}^2\theta\,\bar D^2(D_\alpha V_D W^\alpha)\\
&= \int \mathrm{d}^2\theta\,(\bar D^2 D_\alpha V_D)W^\alpha = -4\int \mathrm{d}^2\theta\,(W_D)_\alpha W^\alpha
\end{aligned}\tag{9.9}$$

where we used $\bar D_{\dot\beta}W^\alpha = 0$ and where the dual W_D is defined from V_D by $(W_D)_\alpha = -\frac{1}{4}\bar D^2 D_\alpha V_D$, as appropriate in the abelian case. Then one can do the functional integral over W and one obtains

$$\int \mathcal{D}V_D \exp\left[\frac{i}{32\pi}\mathrm{Im}\int \mathrm{d}^4x\,\mathrm{d}^2\theta\left(-\frac{1}{\mathcal{F}''(\phi)}W_D^\alpha W_{D\alpha}\right)\right]\ . \tag{9.10}$$

This reexpresses the ($N=1$) supersymmetrized Yang-Mills action in terms of a dual Yang-Mills action with the effective coupling $\tau(a) = \mathcal{F}''(a)$ replaced by $-\frac{1}{\tau(a)}$. Recall that $\tau(a) = \frac{\theta(a)}{2\pi} + \frac{4\pi i}{g^2(a)}$, so that $\tau \to -\frac{1}{\tau}$ generalizes the inversion of the coupling constant discussed in the introduction. Also, it can be shown that the replacement $W \to W_D$ corresponds to replacing $F_{\mu\nu} \to \tilde F_{\mu\nu}$, the electromagnetic dual, so that the manipulations leading to (9.10) constitute a duality transformation that generalizes the old electromagnetic duality of Montonen and Olive. Expressing the $-\frac{1}{\mathcal{F}''(\phi)}$ in terms of ϕ_D one sees from (9.6) that $\mathcal{F}_D''(\phi_D) = -\frac{\mathrm{d}\phi}{\mathrm{d}\phi_D} = -\frac{1}{\mathcal{F}''(\phi)}$ so that

$$-\frac{1}{\tau(a)} = \tau_D(a_D)\ . \tag{9.11}$$

The whole action can then equivalently be written as

$$\frac{1}{16\pi}\mathrm{Im}\int \mathrm{d}^4x\left[\frac{1}{2}\int \mathrm{d}^2\theta\,\mathcal{F}_D''(\phi_D)W_D^\alpha W_{D\alpha} + \int \mathrm{d}^2\theta\,\mathrm{d}^2\bar\theta\,\phi_D^+\mathcal{F}_D'(\phi_D)\right]\ . \tag{9.12}$$

9.2.2 The duality group

To discuss the full group of duality transformations of the action it is most convenient to write it as

$$\frac{1}{16\pi}\mathrm{Im}\int \mathrm{d}^4x\,\mathrm{d}^2\theta\,\frac{\mathrm{d}\phi_D}{\mathrm{d}\phi}W^\alpha W_\alpha + \frac{1}{32i\pi}\int \mathrm{d}^4x\,\mathrm{d}^2\theta\,\mathrm{d}^2\bar\theta\left(\phi^+\phi_D - \phi_D^+\phi\right)\ . \tag{9.13}$$

While we have shown in the previous subsection that there is a duality symmetry

$$\begin{pmatrix} \phi_D \\ \phi \end{pmatrix} \to \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \phi_D \\ \phi \end{pmatrix} , \tag{9.14}$$

the form (9.13) shows that there also is a symmetry

$$\begin{pmatrix} \phi_D \\ \phi \end{pmatrix} \to \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \phi_D \\ \phi \end{pmatrix} \quad , \quad b \in \mathbf{Z} . \tag{9.15}$$

Indeed, in (9.13) the second term remains invariant since b is real, while the first term gets shifted by

$$\frac{b}{16\pi} \mathrm{Im} \int \mathrm{d}^4x \, \mathrm{d}^2\theta \, W^\alpha W_\alpha = -\frac{b}{16\pi} \int \mathrm{d}^4x \, F_{\mu\nu} \tilde{F}^{\mu\nu} = -2\pi b \nu \tag{9.16}$$

where $\nu \in \mathbf{Z}$ is the instanton number. Since the action appears as e^{iS} in the functional integral, two actions differing only by $2\pi\mathbf{Z}$ are equivalent, and we conclude that(9.15) with integer b is a symmetry of the effective action. The transformations (9.14) and (9.15) together generate the group $Sl(2, \mathbf{Z})$. This is the group of duality symmetries.

Note that the metric (9.3) on moduli space can be written as

$$\mathrm{d}s^2 = \mathrm{Im} \, (\mathrm{d}a_D \mathrm{d}\bar{a}) = \frac{i}{2} (\mathrm{d}a \mathrm{d}\bar{a}_D - \mathrm{d}a_D \mathrm{d}\bar{a}) \tag{9.17}$$

where $\langle z_{\mathrm{D}} \rangle = \frac{1}{2} a_D \sigma_3$ and $a_D = \partial \mathcal{F}(a)/\partial a$, and that this metric obviously also is invariant under the duality group $Sl(2, \mathbf{Z})$

9.2.3 Monopoles, dyons and the BPS mass spectrum

At this point, I will have to add a couple of ingredients without much further justification and refer the reader to the literature for more details.

In a spontaneously broken gauge theory as the one we are considering, typically there are solitons (static, finite-energy solutions of the equations of motion) that carry magnetic charge and behave like non-singular magnetic monopoles (for a pedagogical treatment, see Coleman's lectures). The duality transformation (9.14) constructed above exchanges electric and magnetic degrees of freedom, hence electrically charged states, as would be described by hypermultiplets of our $N = 2$ supersymmetric version, with magnetic monopoles.

As for any theory with extended supersymmetry, there are long and short (BPS) multiplets in the present $N = 2$ theory. small (or short) multiplets have

4 helicity states and large (or long) ones have 16 helicity states. As discussed earlier, massless states must be in short multiplets, while massive states are in short ones if they satisfy the BPS condition $m^2 = 2|Z|^2$, or in long ones if $m^2 > 2|Z|^2$. Here Z is the central charge of the $N = 2$ susy algebra rescaled by a factor of $\sqrt{2}$ with respect to our earlier conventions of section 2 (in order to conform with the normalisation used by Seiberg and Witten). The states that become massive by the Higgs mechanism must be in short multiplets since they were before the symmetry breaking and the Higgs mechanism cannot generate the missing $16 - 4 = 12$ helicity states. The heavy gauge bosons[2] have masses $m = \sqrt{2}|a| = \sqrt{2}|Z|$ and hence $Z = a$. This generalises to all purely electrically charged states as $Z = an_e$ where n_e is the (integer) electric charge. Duality then implies that a purely magnetically charged state has $Z = a_D n_m$ where n_m is the (integer) magnetic charge. A state with both types of charge, called a dyon, has $Z = an_e + a_D n_m$ since the central charge is additive. All this applies to states in short multiplets, so-called BPS-states. The mass formula for these states then is

$$m^2 = 2|Z|^2 \quad , \quad Z = (n_m, n_e) \begin{pmatrix} a_D \\ a \end{pmatrix} . \tag{9.18}$$

It is clear that under a $Sl(2, \mathbf{Z})$ transformation $M = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in Sl(2, \mathbf{Z})$ acting on $\begin{pmatrix} a_D \\ a \end{pmatrix}$, the charge vector gets transformed to $(n_m, n_e)M = (n'_m, n'_e)$ which are again integer charges. In particular, one sees again at the level of the charges that the transformation (9.14) exchanges purely electrically charged states with purely magnetically charged ones. It can be shown that precisely those BPS states are stable for which n_m and n_e are relatively prime, i.e. for stable states $(n_m, n_e) \neq (qm, qn)$ for integer m, n and $q \neq \pm 1$.

9.3 Singularities and Monodromy

In this section we will study the behaviour of $a(u)$ and $a_D(u)$ as u varies on the moduli space $\mathcal{M}$. Particularly useful information will be obtained from their behaviour as u is taken around a closed contour. If the contour does not encircle certain singular points to be determined below, $a(u)$ and $a_D(u)$ will return to their initial values once u has completed its contour. However, if the u-contour goes around these singular points, $a(u)$ and $a_D(u)$ do not return to their initial values but rather to certain linear combinations thereof: one has a non-trivial monodromy for the multi-valued functions $a(u)$ and $a_D(u)$.

[2] Again, to conform with the Seiberg-Witten normalisation, we have absorbed a factor of g into a and a_D, so that the masses of the heavy gauge bosons now are $m = \sqrt{2}|a|$ rather than $\sqrt{2}g|a|$.

9.3.1 The monodromy at infinity

This is immediately clear from the behaviour near $u = \infty$. As already explained in section 3.4, as $u \to \infty$, due to asymptotic freedom, the perturbative expression for $\mathcal{F}(a)$ is valid and one has from (9.4) for $a_D = \partial\mathcal{F}(a)/\partial a$

$$a_D(u) = \frac{i}{\pi} a \left(\ln \frac{a^2}{\Lambda^2} + 1 \right) \quad , \quad u \to \infty \, . \tag{9.19}$$

Now take u around a counterclockwise contour of very large radius in the complex u-plane, often simply written as $u \to e^{2\pi i} u$. This is equivalent to having u encircle the point at ∞ on the Riemann sphere in a *clockwise* sense. In any case, since $u = \frac{1}{2} a^2$ (for $u \to \infty$) one has $a \to -a$ and

$$a_D \to \frac{i}{\pi} (-a) \left(\ln \frac{e^{2\pi i} a^2}{\Lambda^2} + 1 \right) = -a_D + 2a \tag{9.20}$$

or

$$\begin{pmatrix} a_D(u) \\ a(u) \end{pmatrix} \to M_\infty \begin{pmatrix} a_D(u) \\ a(u) \end{pmatrix} \quad , \quad M_\infty = \begin{pmatrix} -1 & 2 \\ 0 & -1 \end{pmatrix} \, . \tag{9.21}$$

Clearly, $u = \infty$ is a branch point of $a_D(u) \sim \frac{i}{\pi} \sqrt{2u} \left(\ln \frac{u}{\Lambda^2} + 1 \right)$. This is why this point is referred to as a singularity of the moduli space.

9.3.2 How many singularities?

Can $u = \infty$ be the only singular point? Since a branch cut has to start and end somewhere, there must be at least one other singular point. Following Seiberg and Witten, I will argue that one actually needs three singular points at least. To see why two cannot work, let's suppose for a moment that there are only two singularities and show that this leads to a contradiction.

Before doing so, let me note that there is an important so-called $U(1)_R$-symmetry in the classical theory that takes $z \to e^{2i\alpha} z$, $\phi \to e^{2i\alpha} \phi$, $W \to e^{i\alpha} W$, $\theta \to e^{i\alpha} \theta$, $\bar\theta \to e^{i\alpha} \bar\theta$, thus $d^2\theta \to e^{-2i\alpha} d^2\theta$, $d^2\bar\theta \to e^{-2i\alpha} d^2\bar\theta$. Then the classical action is invariant under this global symmetry. More generallly, the action will be invariant if $\mathcal{F}(z) \to e^{4i\alpha} \mathcal{F}(z)$. This symmetry is broken by the one-loop correction and also by instanton contributions. The latter give corrections to $\mathcal{F}$ of the form $z^2 \sum_{k=1}^\infty c_k \left(\Lambda^2/z^2 \right)^{2k}$, and hence are invariant only for $\left(e^{4i\alpha} \right)^{2k} = 1$, i.e. $\alpha = \frac{2\pi n}{8}$, $n \in \mathbf{Z}$. Hence instantons break the $U(1)_R$-symmetry to a dicrete $\mathbf{Z}_8$. The one-loop corrections behave as $\frac{i}{2\pi} z^2 \ln \frac{z^2}{\Lambda^2} \to e^{4i\alpha} \left(\frac{i}{2\pi} z^2 \ln \frac{z^2}{\Lambda^2} - \frac{2\alpha}{\pi} z^2 \right)$. As before one shows that this only changes the action by $2\pi\nu \left(\frac{4\alpha}{\pi} \right)$ where ν is integer, so that again this change is irrelevant as long as $\frac{4\alpha}{\pi} = n$ or $\alpha = \frac{2\pi n}{8}$. Under this

$\mathbf{Z}_8$-symmetry, $z \to e^{i\pi n/2} z$, i.e. for odd n one has $z^2 \to -z^2$. The non-vanishing expectation value $u = \langle \text{tr } z^2 \rangle$ breaks this $\mathbf{Z}_8$ further to $\mathbf{Z}_4$. Hence for a given vacuum, i.e. a given point on moduli space there is only a $\mathbf{Z}_4$-symmetry left from the $U(1)_R$-symmetry. However, on the manifold of all possible vacua, i.e. on $\mathcal{M}$, one has still the full $\mathbf{Z}_8$-symmetry, taking u to $-u$.

Due to this global symmetry $u \to -u$, singularities of $\mathcal{M}$ should come in pairs: for each singularity at $u = u_0$ there is another one at $u = -u_0$. The only fixed points of $u \to -u$ are $u = \infty$ and $u = 0$. We have already seen that $u = \infty$ is a singular point of $\mathcal{M}$. So if there are only two singularities the other must be the fixed point $u = 0$.

If there are only two singularities, at $u = \infty$ and $u = 0$, then by contour deformation ("pulling the contour over the back of the sphere")[3] one sees that the monodromy around 0 (in a counterclockwise sense) is the same as the above monodromy around ∞: $M_0 = M_\infty$. But then a^2 is not affected by any monodromy and hence is a good global coordinate, so one can take $u = \frac{1}{2}a^2$ on all of $\mathcal{M}$, and furthermore one must have

$$\begin{aligned} a_D &= \tfrac{i}{\pi} a \left(\ln \tfrac{a^2}{\Lambda^2} + 1 \right) + g(a) \\ a &= \sqrt{2u} \end{aligned} \tag{9.22}$$

where $g(a)$ is some entire function of a^2. This implies that

$$\tau = \frac{\mathrm{d}a_D}{\mathrm{d}a} = \frac{i}{\pi} \left(\ln \frac{a^2}{\Lambda^2} + 3 \right) + \frac{\mathrm{d}g}{\mathrm{d}a} \ . \tag{9.23}$$

The function g being entire, $\mathrm{Im}\, \frac{\mathrm{d}g}{\mathrm{d}a}$ cannot have a minimum (unless constant) and it is clear that $\mathrm{Im}\, \tau$ cannot be positive everywhere. As already emphasized, this means that a (or rather a^2) cannot be a good global coordinate and (9.22) cannot hold globally. Hence, two singularities only cannot work.

The next simplest choice is to try 3 singularities. Due to the $u \to -u$ symmetry, these 3 singularities are at ∞, u_0 and $-u_0$ for some $u_0 \neq 0$. In particular, $u = 0$ is no longer a singularity of the quantum moduli space. To get a singularity also at $u = 0$ one would need at least four singularities at $\infty, u_0, -u_0$ and 0. As discussed later, this is not possible, and more generally, exactly 3 singularities seems to be the only consistent possibility.

So there is no singularity at $u = 0$ in the quantum moduli space $\mathcal{M}$. Classically, however, one precisely expects that $u = 0$ should be a singular point, since

[3] It is well-known from complex analysis that monodromies are associated with contours around branch points. The precise from of the contour does not matter, and it can be deformed as long as it does not meet another branch point. Our singularities precisely are the branch points of $a(u)$ or $a_D(u)$.

classically $u = \frac{1}{2}a^2$, hence $a = 0$ at this point, and then there is no Higgs mechanism any more. Thus all (elementary) massive states, i.e. the gauge bosons v_μ^1, v_μ^2 and their susy partners $\psi^1, \psi^2, \lambda^1, \lambda^2$ become massless. Thus the description of the lights fields in terms of the previous Wilsonian effective action should break down, inducing a singularity on the moduli space. As already stressed, this is the clasical picture. While $a \to \infty$ leads to asymptotic freedom and the microscopic SU(2) theory is weakly coupled, as $a \to 0$ one goes to a strong coupling regime where the classical reasoning has no validity any more, and $u \neq \frac{1}{2}a^2$. By the BPS mass formula (9.18) massless gauge bosons still are possible at $a = 0$, but this does no longer correspond to $u = 0$.

So where has the singularity due to massless gauge bosons at $a = 0$ moved to? One might be tempted to think that $a = 0$ now corresponds to the singularities at $u = \pm u_0$, but this is not the case as I will show in a moment. The answer is that the point $a = 0$ no longer belongs to the quantum moduli space (at least not to the component connected to $u = \infty$ which is the only thing one considers). This can be seen explicitly from the form of the solution for $a(u)$ given in the next section.

9.3.3 The strong coupling singularities

Let's now concentrate on the case of three singularities at $u = \infty, u_0$ and $-u_0$. What is the interpretation of the (strong-coupling) singularities at finite $u = \pm u_0$? One might first try to consider that they are still due to the gauge bosons becoming massless. However, as Seiberg and Witten point out, massless gauge bosons would imply an asymptotically conformally invariant theory in the infrared limit and conformal invariance implies $u = \langle \text{tr } z^2 \rangle = 0$ unless tr z^2 has dimension zero and hence would be the unity operator - which it is not. So the singularities at $u = \pm u_0$ ($\neq 0$) do not correspond to massless gauge bosons.

There are no other elementary $N = 2$ multiplets in our theory. The next thing to try is to consider collective excitations - solitons, like the magnetic monopoles or dyons. Let's first study what happens if a magnetic monopole of unit magnetic charge becomes massless. From the BPS mass formula (9.18), the mass of the magnetic monopole is

$$m^2 = 2|a_D|^2 \tag{9.24}$$

and hence vanishes at $a_D = 0$. We will see that this produces one of the two stron-coupling singularities. So call u_0 the value of u at whiche a_D vanishes. Magnetic monopoles are described by hypermultiplets H of $N = 2$ susy that couple locally to the dual fields ϕ_D and W_D, just as electrically charged "electrons" would be described by hypermultiplets that couple locally to ϕ and W. So in the dual description we have ϕ_D, W_D and H, and, near u_0, $a_D \sim \langle \phi_D \rangle$ is small. This theory is exactly $N = 2$ susy QED with very light electrons (and a subscript D on every quantity). The latter theory is not asymptotically free, but has a β-function given by

$$\mu \frac{\mathrm{d}}{\mathrm{d}\mu} g_D = \frac{g_D^3}{8\pi^2} \tag{9.25}$$

where g_D is the coupling constant. But the scale μ is proportional to a_D and $\frac{4\pi i}{g_D^2(a_D)}$ is τ_D for $\theta_D = 0$ (of course, super QED, unless embedded into a larger gauge group, does not allow for a non-vanishing theta angle). One concludes that for $u \approx u_0$ or $a_D \approx 0$

$$a_D \frac{\mathrm{d}}{\mathrm{d}a_D} \tau_D = -\frac{i}{\pi} \ \Rightarrow\ \tau_D = -\frac{i}{\pi} \ln a_D \ . \tag{9.26}$$

Since $\tau_D = \frac{\mathrm{d}(-a)}{\mathrm{d}a_D}$ this can be integrated to give

$$a \approx a_0 + \frac{i}{\pi} a_D \ln a_D \qquad (u \approx u_0) \tag{9.27}$$

where we dropped a subleading term $-\frac{i}{\pi} a_D$. Now, a_D should be a good coordinate in the vicinity of u_0, hence depend linearly[4] on u. One concludes

$$a_D \approx c_0(u - u_0) \quad , \quad a \approx a_0 + \frac{i}{\pi} c_0 (u - u_0) \ln(u - u_0) \ . \tag{9.28}$$

From these expressions one immediately reads the monodromy as u turns around u_0 counterclockwise, $u - u_0 \to e^{2\pi i}(u - u_0)$:

$$\begin{pmatrix} a_D \\ a \end{pmatrix} \to \begin{pmatrix} a_D \\ a - 2a_D \end{pmatrix} = M_{u_0} \begin{pmatrix} a_D \\ a \end{pmatrix} \quad , \quad M_{u_0} = \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix} \ . \tag{9.29}$$

To obtain the monodromy matrix at $u = -u_0$ it is enough to observe that the contour around $u = \infty$ is equivalent to a counterclockwise contour of very large radius in the complex plane. This contour can be deformed into a contour encircling u_0 and a contour encircling $-u_0$, both counterclockwise. It follows the factorisation condition on the monodromy matrices[5]

$$M_\infty = M_{u_0} M_{-u_0} \tag{9.30}$$

and hence

$$M_{-u_0} = \begin{pmatrix} -1 & 2 \\ -2 & 3 \end{pmatrix} \ . \tag{9.31}$$

[4] One might want to try a more general dependence like $a_D \approx c_0(u - u_0)^k$ with $k > 0$. This leads to a monodromy in $Sl(2, \mathbf{Z})$ only for integer k. The factorisation condition below, together with the form of $M(n_m, n_e)$ also given below, then imply that $k = 1$ is the only possibility.

[5] There is an ambiguity concerning the ordering of M_{u_0} and M_{-u_0} which will be resolved below.

What is the interpretation of this singularity at $u = -u_0$? To discover this, consider the behaviour under monodromy of the BPS mass formula $m^2 = 2|Z|^2$ with Z given by (9.18), i.e. $Z = (n_m, n_e)\begin{pmatrix} a_D \\ a \end{pmatrix}$. The monodromy transformation $\begin{pmatrix} a_D \\ a \end{pmatrix} \rightarrow M \begin{pmatrix} a_D \\ a \end{pmatrix}$ can be interpreted as changing the magnetic and electric quantum numbers as

$$(n_m, n_e) \rightarrow (n_m, n_e)M \ . \tag{9.32}$$

The state of vanishing mass responsible for a singularity should be invariant under the monodromy, and hence be a left eigenvector of M with unit eigenvalue. This is clearly so for the magnetic monopole: $(1, 0)$ is a left eigenvector of $\begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}$ with unit eigenvalue. This simply reflects that $m^2 = 2|a_D|^2$ is invariant under (9.29). Similarly, the left eigenvector of (9.31) with unit eigenvalue is $(n_m, n_e) = (1, -1)$ This is a dyon. Thus the sigularity at $-u_0$ is interpreted as being due to a $(1, -1)$ dyon becoming massless.

More generally, (n_m, n_e) is the left eigenvector with unit eigenvalue[6] of

$$M(n_m, n_e) = \begin{pmatrix} 1 + 2n_m n_e & 2n_e^2 \\ -2n_m^2 & 1 - 2n_m n_e \end{pmatrix} \tag{9.33}$$

which is the monodromy matrix that should appear for any singularity due to a massless dyon with charges (n_m, n_e). Note that M_∞ as given in (9.21) is not of this form, since it does not correspond to a hypermultiplet becoming massless.

One notices that the relation (9.30) does not look invariant under $u \rightarrow -u$, i.e $u_0 \rightarrow -u_0$ since M_{u_0} and M_{-u_0} do not commute. The apparent contradiction with the $\mathbf{Z}_2$-symmetry is resolved by the following remark. The precise definition of the composition of two monodromies as in (9.30) requires a choice of base-point $u = P$ (just as in the definition of homotopy groups). Using a different base-point, namely $u = -P$, leads to

$$M_\infty = M_{-u_0} M_{u_0} \tag{9.34}$$

instead. Then one would obtain $M_{-u_0} = \begin{pmatrix} 3 & 2 \\ -2 & -1 \end{pmatrix}$, and comparing with (9.33), this would be interpreted as due to a $(1, 1)$ dyon. Thus the $\mathbf{Z}_2$-symmetry $u \rightarrow -u$ on the quantum moduli space also acts on the base-point P, hence exchanging (9.30) and (9.34). At the same time it exchanges the $(1, -1)$ dyon with the $(1, 1)$ dyon.

Does this mean that the $(1, 1)$ and $(1, -1)$ dyons play a privileged role? Actually not. If one first turns k times around ∞, then around u_0, and then k times

[6] Of course, the same is true for any (qn_m, qn_e) with $q \in \mathbf{Z}$, but according to the discussion in section 4.3 on the stability of BPS states, states with $q \neq \pm 1$ are not stable.

around ∞ in the opposite sense, the corresponding monodromy is $M_\infty^{-k} M_{u_0} M_\infty^k = \begin{pmatrix} 1-4k & 8k^2 \\ -2 & 1+4k \end{pmatrix} = M(1,-2k)$ and similarly $M_\infty^{-k} M_{-u_0} M_\infty^k$ $= \begin{pmatrix} -1-4k & 2+8k+8k^2 \\ -2 & 3+4k \end{pmatrix} = M(1,-1-2k)$. So one sees that these monodromies correspond to dyons with $n_m = 1$ and any $n_e \in \mathbf{Z}$ becoming massless. Similarly one has e.g. $M_{u_0}^k M_{-u_0} M_{u_0}^{-k} = M(1-2k,-1)$, etc.

Let's come back to the question of how many singularities there are. Suppose there are p strong coupling singularities at $u_1, u_2, \ldots u_p$ in addition to the one-loop perturbative singularity at $u = \infty$. Then one has a factorisation analogous to (9.30):

$$M_\infty = M_{u_1} M_{u_2} \ldots M_{u_p} \tag{9.35}$$

with $M_{u_i} = M(n_m^{(i)}, n_e^{(i)})$ of the form (9.33). It thus becomes a problem of number theory to find out whether, for given p, there exist solutions to (9.35) with integer $n_m^{(i)}$ and $n_e^{(i)}$. For several low values of $p > 2$ it has been checked that there are no such solutions, and it seems likely that the same is true for all $p > 2$.

9.4 The solution

Recall that our goal is to determine the exact non-perturbative low-energy effective action, i.e. determine the function $\mathcal{F}(z)$ locally. This will be achieved, at least in principle, once we know the functions $a(u)$ and $a_D(u)$, since one then can invert the first to obtain $u(a)$, at least within a certain domain of the moduli space. Substituting this into $a_D(u)$ yields $a_D(a)$ which upon integration gives the desired $\mathcal{F}(a)$.

So far we have seen that $a_D(u)$ and $a(u)$ are single-valued except for the monodromies around ∞, u_0 and $-u_0$. As is well-known from complex analysis, this means that $a_D(u)$ and $a(u)$ are really multi-valued functions with branch cuts, the branch points being ∞, u_0 and $-u_0$. A typical example is $f(u) = \sqrt{u} F(a,b,c;u)$, where F is the hypergeometric function. The latter has a branch cut from 1 to ∞. Similarly, $\sqrt{u}$ has a branch cut from 0 to ∞ (usually taken along the negative real axis), so that $f(u)$ has two branch cuts joining the three singular points $0, 1$ and ∞. When u goes around any of these singular points there is a non-trivial monodromy between $f(u)$ and one other function $g(u) = u^d F(a',b',c';u)$. The three monodromy matrices are in (almost) one-to-one correspondence with the pair of functions $f(u)$ and $g(u)$.

In the physical problem at hand one knows the monodromies, namely

$$M_\infty = \begin{pmatrix} -1 & 2 \\ 0 & -1 \end{pmatrix}, \quad M_{u_0} = \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}, \quad M_{-u_0} = \begin{pmatrix} -1 & 2 \\ -2 & 3 \end{pmatrix} \tag{9.36}$$

and one wants to determine the corresponding functions $a_D(u)$ and $a(u)$. As will be explained, the monodromies fix $a_D(u)$ and $a(u)$ up to normalisation, which will be determined from the known asymptotics (9.19) at infinity.

The precise location of u_0 depends on the renormalisation conditions which can be chosen such that $u_0 = 1$. Assuming this choice in the sequel will simplify somewhat the equations. If one wants to keep u_0, essentially all one has to do is to replace $u \pm 1$ by $\frac{u \pm u_0}{u_0} = \frac{u}{u_0} \pm 1$.

9.4.1 The differential equation approach

Monodromies typically arise from differential equations with periodic coefficients. This is well-known in solid-state physics where one considers a Schrödinger equation with a periodic potential[7]

$$\left[-\frac{\mathrm{d}^2}{\mathrm{d}x^2} + V(x)\right]\psi(x) = 0 \quad , \quad V(x + 2\pi) = V(x) \ . \tag{9.37}$$

There are two independent solutions $\psi_1(x)$ and $\psi_2(x)$. One wants to compare solutions at x and at $x + 2\pi$. Since, due to the periodicity of the potential V, the differential equation at $x + 2\pi$ is exactly the same as at x, the set of solutions must be the same. In other words, $\psi_1(x + 2\pi)$ and $\psi_2(x + 2\pi)$ must be linear combinations of $\psi_1(x)$ and $\psi_2(x)$:

$$\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}(x + 2\pi) = M \begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}(x) \tag{9.38}$$

where M is a (constant) monodromy matrix.

The same situation arises for differential equations in the complex plane with meromorphic coefficients. Consider again the Schrödinger-type equation

$$\left[-\frac{\mathrm{d}^2}{\mathrm{d}z^2} + V(z)\right]\psi(z) = 0 \tag{9.39}$$

with meromorphic $V(z)$, having poles at $z_1, \ldots z_p$ and (in general) also at ∞. The periodicity of the previous example is now replaced by the single-valuedness of $V(z)$ as z goes around any of the poles of V (with $z - z_i$ corresponding roughly to e^{ix}). So, as z goes once around any one of the z_i, the differential equation (9.39 does not change. So by the same argument as above, the two solutions $\psi_1(z)$

[7] The constant energy has been included into the potential, and the mass has been normalised to $\frac{1}{2}$.

and $\psi_2(z)$, when continued along the path surrounding z_i must again be linear combinations of $\psi_1(z)$ and $\psi_2(z)$:

$$\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}\left(z+e^{2\pi i}(z-z_i)\right)=M_i\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}(z) \tag{9.40}$$

with a constant 2×2-monodromy matrix M_i for each of the poles of V. Of course, one again has the factorisation condition (9.35) for M_∞. It is well-known, that non-trivial constant monodromies correspond to poles of V that are at most of second order. In the language of differential equations, (9.39) then only has *regular* singular points.

In our physical problem, the *two* multivalued functions $a_D(z)$ and $a(z)$ have 3 singularities with non-trivial monodromies at $-1,+1$ and ∞. Hence they must be solutions of a second-order differential equation (9.39) with the potential V having (at most) second-order poles precisely at these points. The general form of this potential is[8]

$$V(z)=-\frac{1}{4}\left[\frac{1-\lambda_1^2}{(z+1)^2}+\frac{1-\lambda_2^2}{(z-1)^2}-\frac{1-\lambda_1^2-\lambda_2^2+\lambda_3^2}{(z+1)(z-1)}\right] \tag{9.41}$$

with double poles at $-1,+1$ and ∞. The corresponding residues are $-\frac{1}{4}(1-\lambda_1^2)$, $-\frac{1}{4}(1-\lambda_2^2)$ and $-\frac{1}{4}(1-\lambda_3^2)$. Without loss of generality, I assume $\lambda_i\geq 0$. The corresponding differential equation (9.39) is well-known in the mathematical literature since it can be transformed into the hypergeometric differential equation. The transformation to the standard hypergeometric equation is readily performed by setting

$$\psi(z)=(z+1)^{\frac{1}{2}(1-\lambda_1)}(z-1)^{\frac{1}{2}(1-\lambda_2)}\,f\left(\frac{z+1}{2}\right) . \tag{9.42}$$

One then finds that f satisfies the hypergeometric differential equation

$$x(1-x)f''(x)+[c-(a+b+1)x]f'(x)-abf(x)=0 \tag{9.43}$$

with

$$a=\frac{1}{2}(1-\lambda_1-\lambda_2+\lambda_3)\ ,\quad b=\frac{1}{2}(1-\lambda_1-\lambda_2-\lambda_3)\ ,\quad c=1-\lambda_1\ . \tag{9.44}$$

The solutions of the hypergeometric equation (9.43) can be written in many different ways due to the various identities between the hypergeometric function $F(a,b,c;x)$ and products with powers, e.g. $(1-x)^{c-a-b}F(c-a,c-b,c;x)$, etc. A convenient choice for the two independent solutions is the following

$$\begin{aligned} f_1(x) &= (-x)^{-a}F(a,a+1-c,a+1-b;\tfrac{1}{x})\\ f_2(x) &= (1-x)^{c-a-b}F(c-a,c-b,c+1-a-b;1-x)\ . \end{aligned} \tag{9.45}$$

[8] Additional terms in V that naively look like first-order poles ($\sim\frac{1}{z-1}$ or $\frac{1}{z+1}$) cannot appear since they correspond to third-order poles at $z=\infty$.

f_1 and f_2 correspond to Kummer's solutions denoted u_3 and u_6. The choice of f_1 and f_2 is motivated by the fact that f_1 has simple monodromy properties around $x=\infty$ (i.e. $z=\infty$) and f_2 has simple monodromy properties around $x=1$ (i.e. $z=1$), so they are good candidates to be identified with $a(z)$ and $a_D(z)$.

One can extract a great deal of information from the asymptotic forms of $a_D(z)$ and $a(z)$. As $z\to\infty$ one has $V(z)\sim -\frac{1}{4}\frac{1-\lambda_3^2}{z^2}$, so that the two independent solutions behave asymptotically as $z^{\frac{1}{2}(1\pm\lambda_3)}$ if $\lambda_3\neq 0$, and as $\sqrt{z}$ and $\sqrt{z}\ln z$ if $\lambda_3=0$. Comparing with (9.22) (with $u\to z$) we see that the latter case is realised. Similarly, with $\lambda_3=0$, as $z\to 1$, one has $V(z)\sim -\frac{1}{4}\left(\frac{1-\lambda_2^2}{(z-1)^2}-\frac{1-\lambda_1^2-\lambda_2^2}{2(z-1)}\right)$, where I have kept the subleading term. From the logarithmic asymptotics (9.28) one then concludes $\lambda_2=1$ (and from the subleading term also $-\frac{\lambda_1^2}{8}=\frac{i}{\pi}\frac{c_0}{a_0}$). The $\mathbf{Z}_2$-symmetry ($z\to -z$) on the moduli space then implies that, as $z\to -1$, the potential V does not have a double pole either, so that also $\lambda_1=1$. Hence we conclude

$$\lambda_1=\lambda_2=1\ ,\quad \lambda_3=0\ \Rightarrow\ V(z)=-\frac{1}{4}\frac{1}{(z+1)(z-1)} \tag{9.46}$$

and $a=b=-\frac{1}{2}$, $c=0$. Thus from (9.42) one has $\psi_{1,2}(z)=f_{1,2}\left(\frac{z+1}{2}\right)$. One can then verify that the two solutions

$$\begin{aligned} a_D(u) &= i\psi_2(u)=i\frac{u-1}{2}F\left(\tfrac{1}{2},\tfrac{1}{2},2;\tfrac{1-u}{2}\right)\\ a(u) &= -2i\psi_1(u)=\sqrt{2}(u+1)^{\frac{1}{2}}F\left(-\tfrac{1}{2},\tfrac{1}{2},1;\tfrac{2}{u+1}\right)\end{aligned} \tag{9.47}$$

indeed have the required monodromies (9.36), as well as the correct asymptotics.

It might look as if we have not used the monodromy properties to determine a_D and a and that they have been determined only from the asymptotics. This is not entirely true, of course. The very fact that there are non-trivial monodromies only at $\infty,+1$ and -1 implied that a_D and a must satisfy the second-order differential equation (9.39) with the potential (9.41). To determine the λ_i we then used the asymptotics of a_D and a. But this is (almost) the same as using the monodromies since the latter were obtained from the asymptotics.

Using the integral representation of the hypergeometric function, the solution (9.47) can be nicely rewritten as

$$a_D(u)=\frac{\sqrt{2}}{\pi}\int_1^u\frac{\mathrm{d}x\ \sqrt{x-u}}{\sqrt{x^2-1}}\ ,\quad a(u)=\frac{\sqrt{2}}{\pi}\int_{-1}^1\frac{\mathrm{d}x\ \sqrt{x-u}}{\sqrt{x^2-1}}\ . \tag{9.48}$$

One can invert the second equation (9.47) to obtain $u(a)$, within a certain domain, and insert the result into $a_D(u)$ to obtain $a_D(a)$. Integrating with respect to a yields $\mathcal{F}(a)$ and hence the low-energy effective action. I should stress that this expression for $\mathcal{F}(a)$ is not globally valid but only on a certain portion of the moduli space. Different analytic continuations must be used on other portions.

9.4.2 The approach using elliptic curves

In their paper, Seiberg and Witten do not use the differential equation approach just described, but rather introduce an auxiliary construction: a certain elliptic curve by means of which two functions with the correct monodromy properties are constructed. I will not go into details here, but simply sketch this approach.

To motivate their construction *a posteriori*, we notice the following: from the integral representation (9.48) it is natural to consider the complex x-plane. More precisely, the integrand has square-root branch cuts with branch points at $+1, -1, u$ and ∞. The two branch cuts can be taken to run from -1 to $+1$ and from u to ∞. The Riemann surface of the integrand is two-sheeted with the two sheets connected through the cuts. If one adds the point at infinity to each of the two sheets, the topology of the Riemann surface is that of two spheres connected by two tubes (the cuts), i.e. a torus. So one sees that the Riemann surface of the integrand in (9.48) has genus one. This is the elliptic curve considered by Seiberg and Witten.

As is well-known, on a torus there are two independent non-trivial closed paths (cycles). One cycle (γ_2) can be taken to go once around the cut $(-1, 1)$, and the other cycle (γ_1) to go from 1 to u on the first sheet and back from u to 1 on the second sheet. The solutions $a_D(u)$ and $a(u)$ in (9.48) are precisely the integrals of some suitable differential λ along the two cycles γ_1 and γ_2:

$$a_D = \oint_{\gamma_1} \lambda \quad , \quad a = \oint_{\gamma_2} \lambda \quad , \quad \lambda = \frac{\sqrt{2}}{2\pi} \frac{\sqrt{x-u}}{\sqrt{x^2-1}} \mathrm{d}x \ . \tag{9.49}$$

These integrals are called period integrals. They are known to satisfy a second-order differential equation, the so-called Picard-Fuchs equation, that is nothing else than our Schrödinger-type equation (9.39) with V given by (9.46).

How do the monodromies appear in this formalism? As u goes once around $+1, -1$ or ∞, the cycles γ_1, γ_2 are changed into linear combinations of themselves with integer coefficients:

$$\begin{pmatrix} \gamma_1 \\ \gamma_2 \end{pmatrix} \to M \begin{pmatrix} \gamma_1 \\ \gamma_2 \end{pmatrix} \quad , \quad M \in Sl(2, \mathbf{Z}) \ . \tag{9.50}$$

This immediately implies

$$\begin{pmatrix} a_D \\ a \end{pmatrix} \to M \begin{pmatrix} a_D \\ a \end{pmatrix} \tag{9.51}$$

with the same M as in (9.50). The advantage here is that one automatically gets monodromies with *integer* coefficients. The other advantage is that

$$\tau(u) = \frac{\mathrm{d}a_D/\mathrm{d}u}{\mathrm{d}a/\mathrm{d}u} \tag{9.52}$$

can be easily seen to be the τ-parameter describing the complex structure of the torus, and as such is garanteed to satisfy $\mathrm{Im}\,\tau(u) > 0$ which was the requirement for positivity of the metric on moduli space.

To motivate the appearance of the genus-one elliptic curve (i.e. the torus) *a priori* - without knowing the solution (9.48) from the differential equation approach - Seiberg and Witten remark that the three monodromies are all very special: they do not generate all of $Sl(2,\mathbf{Z})$ but only a certain subgroup $\Gamma(2)$ of matrices in $Sl(2,\mathbf{Z})$ congruent to 1 modulo 2. Furthermore, they remark that the u-plane with punctures at $1, -1, \infty$ can be thought of as the quotient of the upper half plane H by $\Gamma(2)$, and that $H/\Gamma(2)$ naturally parametrizes (i.e. is the moduli space of) elliptic curves described by

$$y^2 = (x^2-1)(x-u)\ . \tag{9.53}$$

Equation (9.53) corresponds to the genus-one Riemann surface discussed above, and it is then natural to introduce the cycles γ_1, γ_2 and the differential λ from (9.48). The rest of the argument then goes as I just exposed.

9.5 Summary

We have seen realised a version of electric-magnetic duality accompanied by a duality transformation on the expectation value of the scalar (Higgs) field, $a \leftrightarrow a_D$. There is a manifold of inequivalent vacua, the moduli space $\mathcal{M}$, corresponding to different Higgs expectation values. The duality relates strong coupling regions in $\mathcal{M}$ to the perturbative region of large a where the effective low-energy action is known asymptotically in terms of $\mathcal{F}$. Thus duality allows us to determine the latter also at strong coupling. The holomorphicity condition from $N=2$ supersymmetry then puts such strong constraints on $\mathcal{F}(a)$, or equivalently on $a_D(u)$ and $a(u)$ that the full functions can be determined solely from their asymptotic behaviour at the strong and weak coupling singularities of $\mathcal{M}$.

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